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Introducing Parametric Equations through Graphing Calculator Explorations

The Algebra Standard of *Principles and Standards for School Mathematics* (NCTM 2000) suggests that the mathematics curriculum for grades 9–12 include the use of "a variety of symbolic representations, including recursive and parametric equations, for functions and relations" (p. 296). Parametric equations are very useful for representing graphs of curves that cannot otherwise be expressed as functions that define y in terms of x. The underlying idea of working with parametric equations is to express both x and y as functions of a third variable, called the parameter. The parametric equations are those functions assigned to x and y. The variable, typically t for time, makes parametric equations practical for modeling situations involving motion of an object along a given path by providing the coordinates of positions (x, y) of the object over time.

WORKSHEET 1: THE BUG RACE

In order to guide students in developing the algebraic representation that fits the scenario in worksheet 1, it may be necessary to provide the general form of the distance equation, $D = R \cdot T$, where D represents distance traveled, R represents rate, and T represents time. This formula can serve as a discussion platform for determining the necessary parameters to model the movement of Bug 1 and Bug 2. Letting T represent the travel time of Bug 1 in this scenario, the distance traveled by each bug can then be modeled by $D_{\text{Bug 1}} = 12T$ and $D_{\text{Bug 2}} = 18(T - 1)$. On the graphing calculator, the distance traveled by the bugs is modeled in the horizontal direction, with the distance equations stored in functions assigned to x. In this case the functions for the corresponding y-coordinates need only use a constant to model the bugs proceeding on parallel linear paths.

Students need to have calculators in parametric, simultaneous mode. After setting these appropriate mode options, students can then press Y= and notice that the calculator displays a setup that allows the students to input both components of the parametric model for each bug. Pressing the X,T,θ,n button that normally produces X in function mode now automatically produces T. **Figure 1** shows the mode, parametric equation setup, and an appropriate window for this scenario. Upon pressing GRAPH, students can watch the parametric representation as Bug 1 begins moving along the upper path, followed by Bug 2 beginning its movement on the lower path a short while later. Pressing ENTER pauses the plotting of a graph so that students can estimate that Bug 2 draws alongside and passes Bug 1 somewhere between distances $x = 35$ and $x = 40$. Students can compare this visual information to numeric information accessed by pressing 2nd TABLE and scrolling through values to determine that both Bug 1 and Bug 2 are at a distance of $x = 36$ inches from their respective starting points at the same time (**fig. 2**). Students may also use the TRACE feature on the completed graph (**fig. 3**).
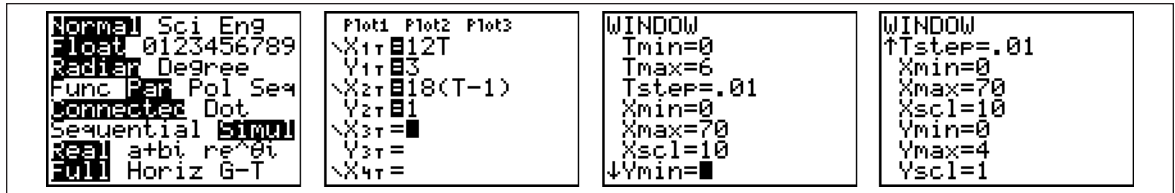
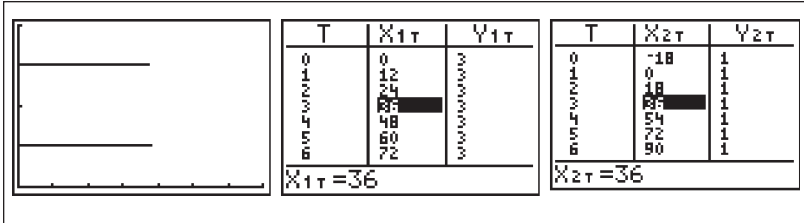**Fig. 1** Calculator settings for the bug race

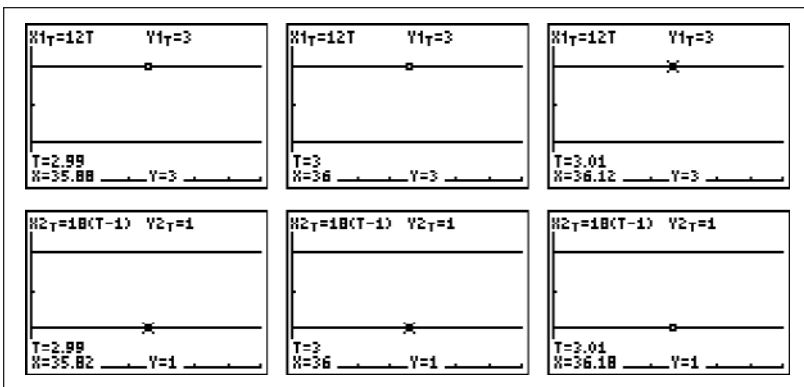**Fig. 2** The bugs are side by side 36 inches from the starting point.

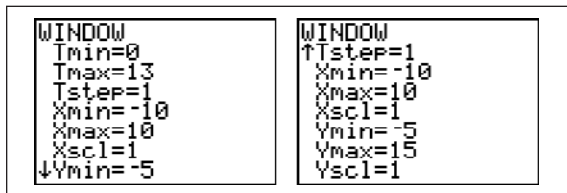**Fig. 3** The trace function can also be used to determine when the bugs are alongside each other.

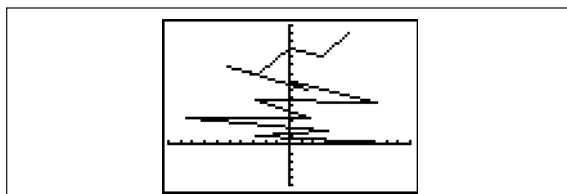**Fig. 4** Window for the nonlinear path of the bug in worksheet 2

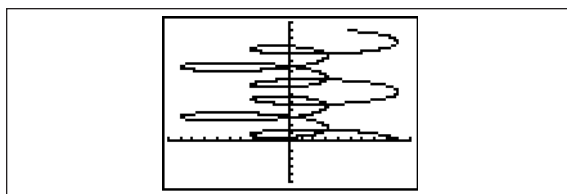**Fig. 5** Calculator graph of the bug in worksheet 2 with a *t*-step of 1

Fig. 6 Graph of the same bug in worksheet 2 with a *t*-step of 0.1

By setting the equations equal to each other to solve for the time at which the bugs are at exactly the same distance from their starting points, students can see how one could view this situation from a purely symbolic format. Students will make connections between previously addressed processes and new concepts involving parametric equations and will begin to gain a sense of the power of parametric modeling with this problem.

WORKSHEET 2: NONLINEAR PATH OF A BUG'S MOTION

The second worksheet considers a bug's path as the bug moves on the *xy* plane in time *t* seconds as modeled by

$$\begin{cases} x = 4\cos(t) + 5\cos(3t) \\ y = \sin(3t) + t \end{cases}$$

To answer the first question on the student worksheet, students can use $t = 0$ to represent the start of the problem and thus substitute 0 for *t* in the *x*- and *y*- components of the model. Since $4\cos(0) + 5\cos(3 \cdot 0) = 9$ and $\sin(3 \cdot 0) + 0 = 0$, the bug starts at the point (9, 0) in the *xy* plane. Students create a table and plot points showing the bug's position and connect the points to create a rudimentary path of the bug's motion. **Figure 5** shows the resulting graph of the bug's motion.

Having students complete the graph by hand makes it easier for them to choose window values on the calculator. I ask students to press WINDOW and consider appropriate values in order to create the same graph that they created by hand. Thus, they set the window values shown in **figure 4**.

When the "bug" stops moving, the resulting graph is the same as **figure 5**. My students have commented that the graph looks a bit "jiggity-jaggity" and have wondered if the graph could be "smoothed out." Such questions from students excite me because they demonstrate interest in the problem and open avenues for further investigation. Through experimentation with their calculators, students find that both changing ΔTbl in table settings to 0.1 and changing the Tstep of the window settings to 0.1 yield a graph smoother than the original. Changing the same two settings to 0.01 yields even more information about the bug's location over the same 13 seconds and noticeably slows

down the process of plotting the graph. The graph with a time increment setting of 0.1 is shown in **figure 6**.

WORKSHEET 3: PATHS OF THREE BUGS' MOTION

After completing worksheets 1 and 2, my students are usually excited to see the paths the three bugs leave on the xy plane. Thus, I leave it to them to experiment with their calculators to figure out which path on a given static graph (**fig. 7**) belongs to which bug, asking students to label each path in the given graph with an A, B, or C, as appropriate. **Figure 8** shows calculator settings that match the given graph well.

Is there a way to tell which graph matches which bug without doing the graphing? Students should look for similarities and differences in the models. The models for Bug B and Bug C look similar and both produce straight lines as opposed to the curve produced by the model for Bug A. I take advantage of students' thoughts and questions to introduce the idea that parametric equations can be rewritten as relationships involving only x- and y-variables (without the parameter of time) through algebraic substitution. For example, using the model for Bug B, we can rewrite $x = 3t$ as $t = x/3$ and then replace t with $x/3$ in $y = 5t$ to end up with $y = 5(x/3)$ or $y = (5/3)x$. Similarly, the model for

Bug C can be rewritten as $y = (4/3)x$. Then, using what students already know about slopes of lines, they can determine that the steeper of the two lines provided in the graph must belong to Bug B. The remaining path, the path of Bug A, is part of an ellipse generated by its given model, which can be expressed by $y = 5\sin(\cos^{-1}(x/8))$.

Figure 9 displays the setup needed to plot Bug A's and Bug B's paths simultaneously. The resulting plots in motion show that Bug B gets out of Bug A's path before Bug A reaches the static graph's point of intersection.

Students can quickly determine that Bug B and Bug C do not collide after leaving the same starting point, because of the nature of their linear paths at different slopes. On the other hand, it takes further investigation to determine whether or not Bug A and Bug C collide, and students may need to turn to other methods for an answer. Some of my students have used the table, scrolling to look for matching position values for A and C. Note that this can be done in pairs, with one student viewing A's position values while the other student views those of C, since the calculator does not show all values on the same screen (**fig. 10**). In fact, at $t = 1.13$ seconds, the bugs' positions are very close. In this way, some students claim that Bug A and Bug C collide at $t = 1.13$ seconds.

It is interesting to note that the calculator features do not permit finding points of intersection in parametric mode as can be done in function mode. Switching back to parametric mode and scrolling through the table to look for this point reveals that Bug A arrived at the point at 1.131 seconds and Bug C arrived at the point at 1.133 seconds. Thus, students can conclude with some justification that Bug A and Bug C do collide at $t = 1.13$ seconds.
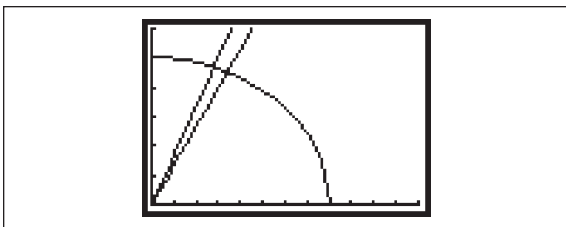

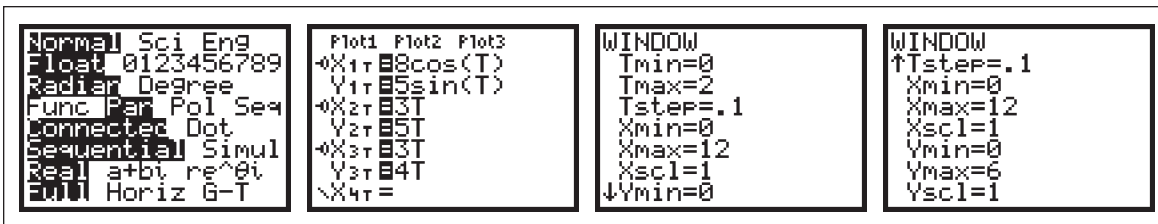**Fig. 7** Which graph corresponds to which equation in figure 8?

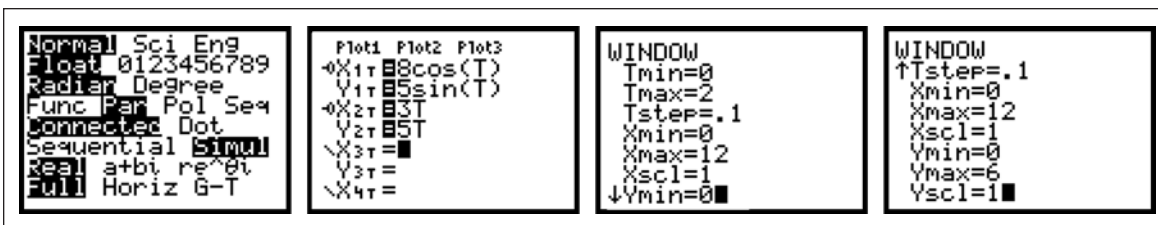**Fig. 8** Calculator settings for the graph in figure 7

Fig. 9 Calculator settings for graphing Bug A and Bug B simultaneously

Fig. 10 The calculator's screen is not wide enough to see more than these three columns.

Fig. 11 Table for bug positions at times t = 1, 2, 3, . . . , 13

SUMMARY

Heid et al. (1995) summarize that graphing technologies allow a ready visualization of relationships; allow for finding accurate solutions that may not be easily found through symbolic manipulation alone; provide numerical and graphical solutions that support results found using algebraic manipulation; promote exploration by students and enhance their understanding; encourage the exploration of relationships and mathematical concepts; and promote "what if" modeling of realistic situations. These worksheets allow students to explore the relationship of an object's position with respect to time. Students are thus able to recognize the benefit of using parametric equations to determine not only where an object has been, but also when the object was at a given position on its path. The three problems described above also incorporate many of the advantages of integrating the use of graphing calculators as an introduction to parametric equations. The graphing calculator technology makes the problems accessible, understandable, and even fun for students at the 9–12 grade level.

SOLUTIONS
Worksheet 1
1. $12T$ and $18(T-1)$
2. student hypothesis
3. horizontal parallel paths along $y = 1$ and $y = 3$
4. Bug 2
5. $t = 3$ hours, distance, $x = 36$ inches from the start

Worksheet 2
1. (9, 0)
2. (–2.79, 1.14)
3. See **figure 11**.
4. See **figure 4**.
5. See **figure 6**.
6. The added points create a smoother graph.

Worksheet 3
1. B steepest linear path, A curved
2. No. See notes.
3. No, nature of linear paths.
4. Yes. See notes.

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Hungerford, Thomas W. *Contemporary Precalculus: A Graphing Approach.* 3rd ed. Orlando, FL: Harcourt College Publishers, 2000.

Larson, Ron, Robert P. Hostetler, and Bruce H. Edwards. *Precalculus with Limits: A Graphing Approach*. 4th ed. Boston, MA: Houghton Mifflin, 2005.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000. ∞

MARLENA HERMAN is an assistant professor in the Department of Mathematics at Rowan University, Glassboro, NJ 08028. She has taught high school mathematics for five years and college-level mathematics for seven years. Her research interests revolve around the use of technology in teaching and learning mathematics as well as the teaching and learning of mathematics at K–12 levels.

Introducing Parametric Equations through Explorations Sheet 1

Suppose two bugs are crawling along linear paths. Bug 1 begins a trek toward a point 70 inches from where he begins, traveling at a speed of 12 inches per hour. Bug 2 travels at a speed of 18 inches per hour but leaves 1 hour after the other bug from a similar starting position on a parallel path. Note: distance = rate • time (D = R • T).

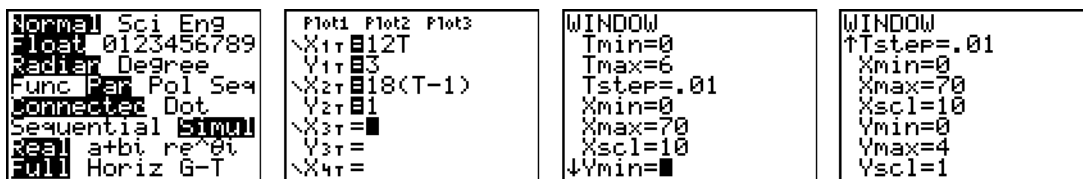1. Given that T represents Bug 1's travel time, what formulas represent the distance each bug travels over time?

Distance$_{Bug\ 1}$ = _____ • _____
                   Rate   Time

Distance$_{Bug\ 2}$ = _____ • _____
                   Rate   Time

2. Which bug do you think will win the race? Why?

Let's watch the race!

```
Normal Sci Eng        Plot1 Plot2 Plot3    WINDOW             WINDOW
Float 0123456789      \X1T∎12T             Tmin=0            ↑Tstep=.01
Radian Degree          Y1T∎3               Tmax=6             Xmin=0
Func Par Pol Seq      \X2T∎18(T-1)         Tstep=.01          Xmax=70
Connected Dot          Y2T∎1               Xmin=0             Xscl=10
Sequential Simul      \X3T=∎               Xmax=70            Ymin=0
Real a+bi re^θi        Y3T =               Xscl=10            Ymax=4
Full Horiz G-T        \X4T =              ↓Ymin=∎             Yscl=1
```

3. The Y equations are set to constants to show the bugs crawling across the screen. Why is there no T variable in the Y equation for each bug?

4. Graph the paths of the bugs in motion. Which bug wins the race?

5. At what time are the bugs the same distance from their starting points along their paths? In other words, when are the bugs alongside each other?

Introducing Parametric Equations through Explorations Sheet 2

Consider a bug's nonlinear path as the bug runs around on the xy plane in time t seconds as modeled by

$$\begin{cases} x = 4\cos(t) + 5\cos(3t) \\ y = \sin(3t) + t \end{cases}$$

(Use MODE: RADIAN and PARAMETRIC)

1. Where does the bug start (what are the coordinates of its initial position)?

2. Where is the bug 1 second after starting?

3. Build a table for the bug's location at times $t = 2, 3, 4, \ldots, 13$ seconds.

4. Use the table of values from question 3 to graph the bug's path from point to point, starting at $t = 0$ and ending at $t = 13$.
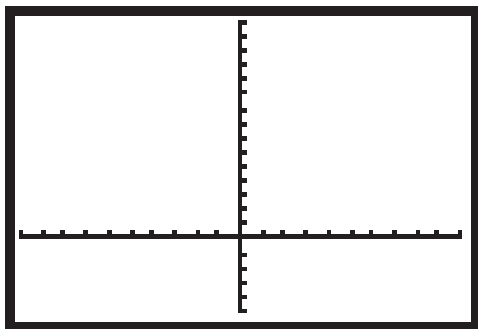
Time	Location	
(sec)	x	y
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		

5. How should we adjust the graph settings if we want to see the bug's path in "slow motion" by looking at where it is every tenth of a second instead of every second? Show the new graph below.
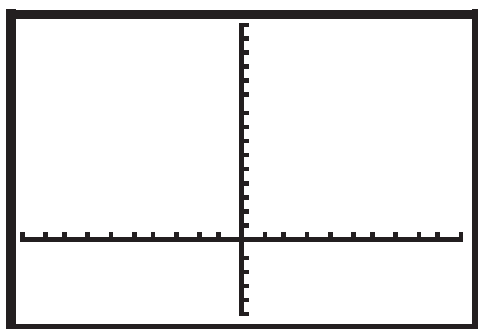
6. How does the graph of the bug's path at every hundredth of a second compare to the graph of the bug's path at every tenth of a second?

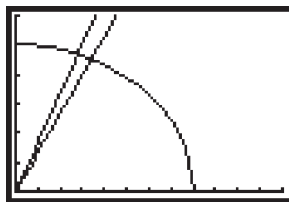# Introducing Parametric Equations through Explorations Sheet 3

Now consider *three* bugs' paths as the bugs run around on the *xy* plane in time *t* seconds as modeled by

Bug A	**Bug B**	**Bug C**
$\begin{cases} x = 8\cos(t) \\ y = 5\sin(t) \end{cases}$	$\begin{cases} x = 3t \\ y = 5t \end{cases}$	$\begin{cases} x = 3t \\ y = 4t \end{cases}$

1. Which path on the graph is Bug A's path? Which is Bug B's path? Bug C's path? Label each path in the picture with an A, B, or C, as appropriate.

   If all three bugs start moving from their initial positions at the same time:

2. Do Bug A and Bug B collide? Justify your answer.

3. Do Bug B and Bug C collide? Justify your answer.

4. Do Bug A and Bug C collide? Justify your answer.