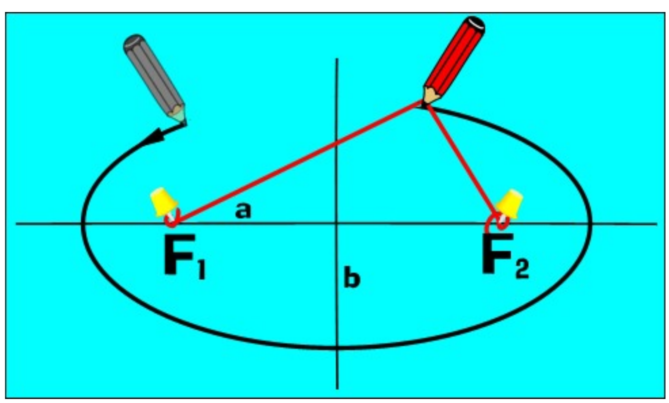
The Focus is the Foci: Ellipses and Hyperbolas

Ellipses and their Foci

The first type of quadratic relation we want to discuss is an ellipse. In terms of its "conic" definition, you can see how a plane would intersect with a cone, making the ellipse below.

We're going to start our study of ellipses by doing a very basic drawing activity. You should have two thumb tacks, a piece of string, a piece of cardboard, and a pencil. Attach the string via the two tacks to the piece of cardboard. Make sure to leave some slack in the string when you pin the ends down so that you can actually draw your outline! Trace out the ellipse by moving the pencil around as far as it will go with the string, making sure that the string is held tight against the pencil.



Of course, you should notice that the shape that results from this construction is an oval. (The term oval is not precise and includes many closed rounded shapes. This particular oval is an ellipse.)

(a) What do the two thumbtacks represent in this activity?

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The two thumbtacks represent foci of the ellipse.

(b) A "locus" of points is a set of points that share a property. Thinking about the simple activity that you just completed, what is the property shared by the entire set of points that make up the ellipse?

An ellipse is the entire set of points on a plane where the sum of the distances from two fixed foci is a constant.

c) Move the string and consider other ellipses created in this manner with different foci. How does the placement of the foci affect the size of the ellipse? How do you know?

When the foci are farther apart, the ellipse is more elongated. When the foci are closer together, the ellipse more closely resembles a circle.

d) What is the length of the string in relation to these ellipses?

The length of the string is the same as the length of the major axis of the ellipse.

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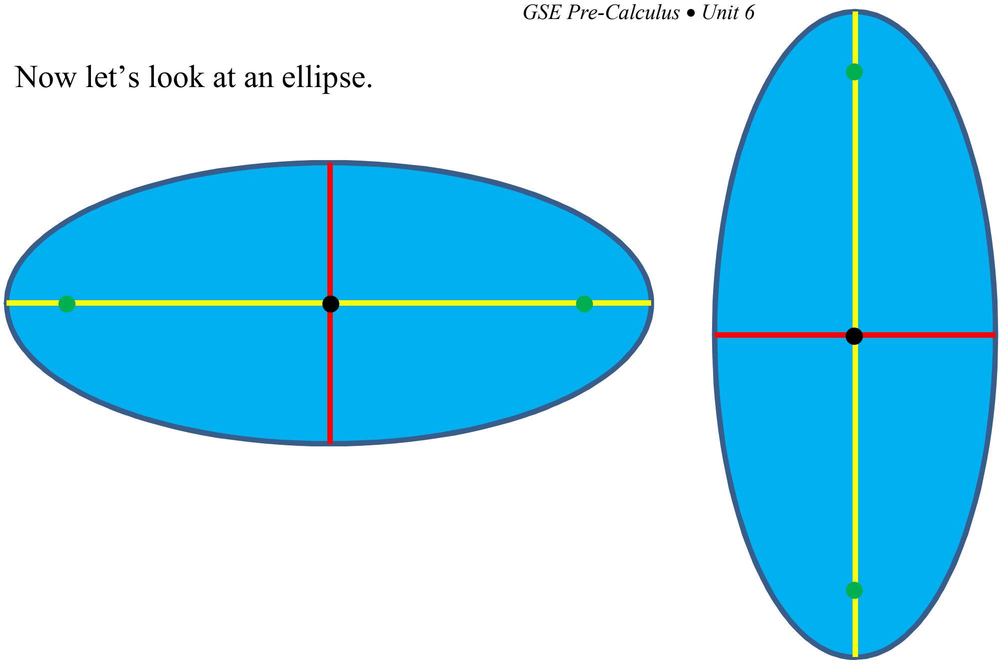
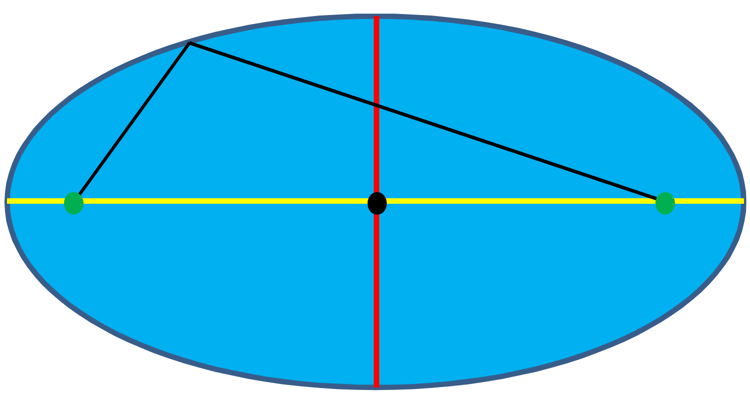
Now let's look at an ellipse.



There are two types of ellipses that we'll be interested in during this unit – a horizontally oriented ellipse (left) and a vertically oriented ellipse (right). (Like all conic sections, these relations can be rotated diagonally if they contain the Bxy term from the general form of a quadratic relation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, but we won't be dealing with these right now – they'll come up in Calculus later on.)

What is the primary difference here? It's a matter of the major axis. Since an ellipse contains two axes of symmetry, the major axis is the longer, and the minor axis is the shorter. The major axis contains the two foci of the ellipse and has vertices as its endpoints. (The endpoints of the minor axis are called the co-vertices.) And speaking of foci, the green dots that you see represent the foci (the thumbtacks you just used), so let's go back and look at how an ellipse can be constructed (or defined) by using its two foci.



What you will hopefully recognize here is that the black lines represent where the string would have been as you were tracing with your pencil. And if you were able to answer (b) above correctly, you already know the relationship that binds the points of an ellipse together – the sum

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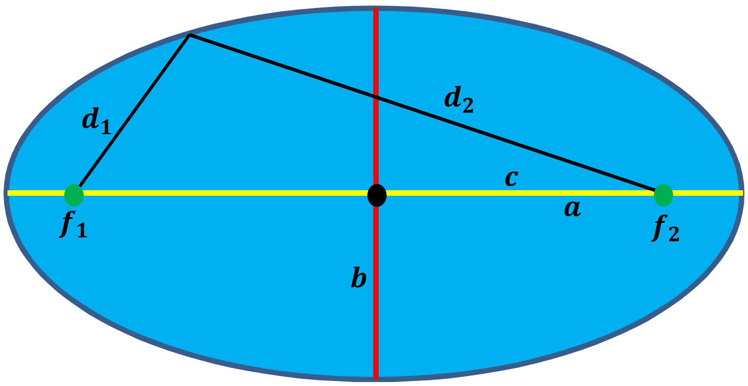
of the distances from each focus to any point on the ellipse remains constant. Another important piece of information that you may have noticed is that when you took your pencil and traced to one of the vertices (endpoints of the major axis), the entire length of string was being used going in a single direction. Therefore, the length of the string ended up being the length of your major axis! So let's fill this ellipse in with some important information.



Let's define what we see in the diagram:

- $\boldsymbol{a}$ is half the length of the major axis
- $\boldsymbol{b}$ is half the length of the minor axis
- $\boldsymbol{c}$ is the length from the center to a focus
- $\boldsymbol{d_1}$ is the distance between the first focus (f_1) and the point of interest
- $\boldsymbol{d_2}$ is the distance between the second focus (f_2) and the point of interest

Using these pieces of an ellipse, we can write out some important facts about this planar curve.

- The length of the major axis is $2a$.
- The length of the minor axis is $2b$.
- The distance between f_1 and f_2 is $2c$.
- $d_1 + d_2 = C$ where $C$is a constant. This is true regardless of the individual values of d_1 and d_2. What is the value of this constant? *2a*

There's one other important relationship among these variables that we need to explore. Remember that we concluded that the length of your string was the length of the major axis?

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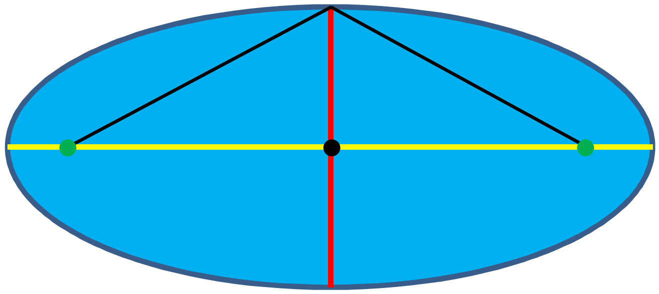
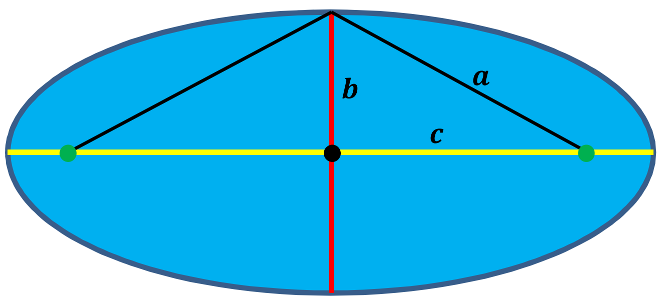
That has some important implications. Let's look at the diagram below.



Here's what we can deduce from the information we have thus far. When the string was in this position, then d_1 and d_2 were of equal length, forming two right triangles with the axes of the ellipse. Since we already know that the entire string length is equal to the major axis length, $2a$, that leads to an important conclusion, namely that $\boldsymbol{d_1 + d_2 = 2a}$. Since d_1 and d_2 are of equal length at this position, that means they both equal a, so we can see the following…



Since we're dealing with a right triangle, the Pythagorean Theorem applies, so

$$\boldsymbol{a^2 = b^2 + c^2}$$

or, to rearrange…

$$\boldsymbol{c^2 = a^2 - b^2}$$

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Geometric Definition of an Ellipse:

The set of all points in a plane such that the sum of the distances from two fixed points (foci) is constant.

Standard Form of an Ellipse with Center $(\boldsymbol{h}, \boldsymbol{k})$:

$Horizontal\ Major\ Axis$: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

$Vertical\ Major\ Axis$: $\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$

where $a > b$ and $c^2 = a^2 - b^2$

Teacher Note: There is quite a jump from the definition of the ellipse to this formula. To help make the connections first either A) derive the formula for the case with center at the origin using the definition and then ask students to translate remembering their translation skills with circles or else B) look at the equation of a circle, divide it by the square of the radius and then ask how we might alter this to allow for different radii in the x and y directions.

Just as with circles and parabolas, we often have to write the equation of an ellipse in standard form (as always, a more useful form) when it is given in another form. And once again, we'll be using the method of completing the square to convert to standard form. For example, if given the equation

$$25x^2 + 9y^2 - 200x + 18y + 184 = 0$$

we should recognize this equation as being in general form. We will now convert to standard form by completing the square.

$$25x^2 - 200x + 9y^2 + 18y = -184$$

$$25(x^2 - 8x) + 9(y^2 + 2y) = -184$$

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$$25\left(x^2 - 8x + \left(\frac{8}{2}\right)^2\right) + 9\left(y^2 + 2y + \left(\frac{2}{2}\right)^2\right) = -184 + 25\left(\frac{8}{2}\right)^2 + 9\left(\frac{2}{2}\right)^2$$

$$25(x-4)^2 + 9(y+1)^2 = 225$$

$$\frac{25(x-4)^2}{225} + \frac{9(y+1)^2}{225} = 1$$

$$\frac{(x-4)^2}{9} + \frac{(y+1)^2}{25} = 1$$

So from our standard form, we know that the center of the ellipse is $(4, -1)$. We know the ellipse has a vertical major axis (since the denominator is larger under the y-term) and we know that $a = 5$ and $b = 3$. Therefore, the vertices would be at $(4, -1+5)$ and $(4, -1-5)$, simplifying to $(4, 4)$ and $(4, -6)$ and the co-vertices would be at $(4+3, -1)$ and $(4-3, -1)$ which simplifies to (7,-1) and (1,-1). To find the coordinates of the foci, we'd do the following:

$$c^2 = a^2 - b^2$$

$$c^2 = 25 - 9 = 16$$

$$c = 4$$

So the foci are at $(4, -1+4)$ and $(4, -1-4)$, simplifying to $(4, 3)$ and $(4, -5)$.

Hyperbolas and their Foci

Although hyperbolas and ellipses are quite different, their formulas and foci relationships are similar, making it is easy to confuse their characteristics. Be careful when working with both of these conics.

We spent quite a bit of time deriving the definition of an ellipse from the relationship of its locus to its foci, and we're going to introduce hyperbolas with the same type of thinking, but we'll be brief.

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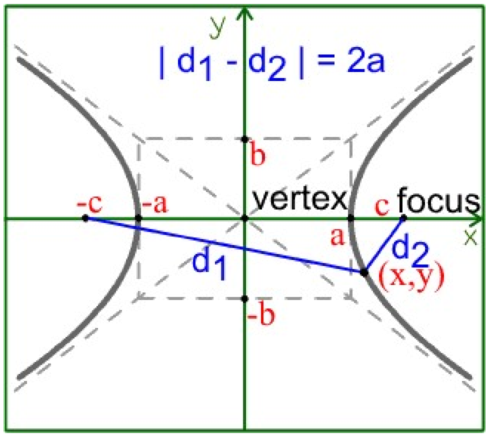

Although on a smaller scale, the graph of the hyperbola looks like an inverted ellipse, on a larger scale, the hyperbola extends forever and approaches 2 intersecting lines called asymptotes. Remember that we define an ellipse as the set of all points on a plane where the sum of the distances from two fixed points called foci is a constant. The relationship of a hyperbola to its foci is slightly different. Notice the blue line segments representing d_1 and d_2. These are no longer being added to obtain a constant – they are now being subtracted!

Geometric Definition of a Hyperbola:

The set of all points in a plane such that the absolute value of the difference of the distances from two fixed points (foci) is constant.

Note that $|d_1 - d_2| = 2a$, and a is still the distance from the center to a vertex, but we no longer call the axis containing the vertices and foci the major axis. We now call the segment joining the vertices the transversal axis, and it no longer must be the longer of the two axes. The other axis of symmetry for a hyperbola contains the conjugate axis, and these 2 axes bisect each other. Here are some other important characteristics of the graphs of hyperbolas:

- The center is the starting point at (h, k).
- The transverse axis contains the foci and the vertices.
- Transverse axis length = $2a$. This is also the constant that the difference of the distances must be.
- Conjugate axis length = $2b$.
- Distance between foci = $2c$.
- The foci are within the curve.
- Since the foci are the farthest away from the center, c is the largest of the three lengths, and the Pythagorean relationship is: $c^2 = a^2 + b^2$.

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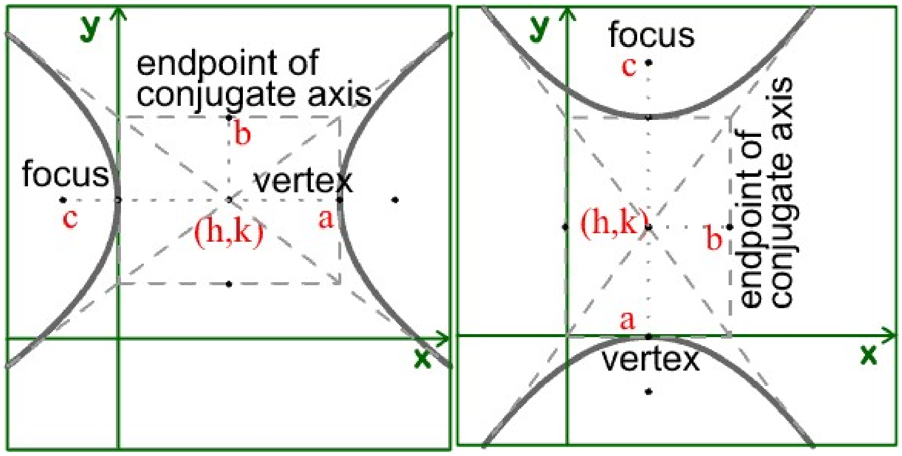
The two types of hyperbolas that we will study are: horizontally (left) and vertically (right) oriented.



The Standard Form of a Hyperbola with Center (h, k):

$Horizontal\ Transverse\ Axis$: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

Since the change in y is b and the change in x is a, the slope of the asymptotes will be $\pm\frac{b}{a}$. The equations of the asymptotes will be $(y-k) = \pm\frac{b}{a}(x-h)$.

$Vertical\ Transverse\ Axis$: $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

Since the change in y is a and the change in x is b, the slope of the asymptotes will be $\pm\frac{a}{b}$. The equations of the asymptotes will be $(y-k) = \pm\frac{a}{b}(x-h)$.

It's probably not surprising that, given the two focal distances are subtracted for hyperbolas instead of added, the standard form of a hyperbola involved subtraction instead of addition (like circles and ellipses).

And just as with circles, parabolas, and ellipses, we sometimes have to take a hyperbola written in another form and convert it to standard form in order to pick out the necessary information to graph the relation accurately. For example, consider the general form of this hyperbola:

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$$9x^2 - 4y^2 + 90x + 32y + 197 = 0$$

$$9x^2 + 90x - 4y^2 + 32y = -197$$

As always, we need to complete the square in order to convert forms.

$$9(x^2 + 10x) - 4(y^2 - 8y) = -197$$

$$9\left(x^2 + 10x + \left(\frac{10}{2}\right)^2\right) - 4\left(y^2 - 8y + \left(\frac{8}{2}\right)^2\right) = -197 + 9\left(\frac{10}{2}\right)^2 - 4\left(\frac{8}{2}\right)^2$$

$$9(x+5)^2 - 4(y-4)^2 = -36$$

$$\frac{9(x+5)^2}{-36} - \frac{4(y-4)^2}{-36} = \frac{-36}{-36}$$

$$-\frac{(x+5)^2}{4} + \frac{(y-4)^2}{9} = 1$$

or

$$\frac{(y-4)^2}{9} - \frac{(x+5)^2}{4} = 1$$

So now we know that $a = 3$ and $b = 2$ and that the center of this hyperbola is $(-5, 4)$ and, since it has a vertical transverse axis, the vertices of the hyperbola are at $(-5, 4+3)$ and $(-5, 4-3)$, which become $(-5, 7)$ and $(-5, 1)$. To find the foci, we use

$$c^2 = a^2 + b^2$$

$$c^2 = 9 + 4 = 13$$

$$c = \sqrt{13}$$

So the foci are at $\left(-5, 4 + \sqrt{13}\right)$ and $\left(-5, 4 - \sqrt{13}\right)$.

So now all we need to do is find the asymptotes. Again, we know that this hyperbola has a vertical transverse axis, and therefore the slope of the asymptotes will be $\pm\frac{\Delta y}{\Delta x} = \pm\frac{3}{2}$. Therefore,

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$$(y-4) = \pm\frac{3}{2}(x-(-5))$$

$$y-4 = \frac{3}{2}x + \frac{15}{2} \text{ and } y-4 = -\frac{3}{2}x - \frac{15}{2}$$

Therefore, the equations of the asymptotes are

$$y = \frac{3}{2}x + \frac{23}{2} \text{ and } y = -\frac{3}{2}x - \frac{7}{2}$$

And now it's your turn…

For the following, put the equation in standard form, label the important pieces, and sketch the graph of the relation.

1. $4x^2 + 9y^2 - 16x + 90y + 205 = 0$

$$\frac{(x-2)^2}{9} + \frac{(y+5)^2}{4} = 1$$

2. $100x^2 + 36y^2 > 3600$

$$\frac{x^2}{36} + \frac{y^2}{100} > 1$$

3. $9x^2 - 4y^2 - 54x - 16y - 79 = 0$

$$\frac{(x-3)^2}{16} - \frac{(y+2)^2}{36} = 1$$

4. $25x^2 - 4y^2 + 200x - 8y + 796 = 0$

$$-\frac{(x+4)^2}{16} + \frac{(y+1)^2}{100} = 1$$

or $\frac{(y+1)^2}{100} - \frac{(x+4)^2}{16} = 1$

5. Write the equation of a hyperbola whose center is at the origin, has a horizontal transverse axis and has asymptotes of $y = \pm\frac{5}{7}x$.

$$\frac{x^2}{49} - \frac{y^2}{25} = 1$$

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6. Write the equation of the ellipse with major axis of length 12 and foci (3, 0) and (-3, 0).

$$c^2 = a^2 - b^2$$

$$9 = 36 - b^2$$

$$27 = b^2$$

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

7. Write the equation of a hyperbola with asymptote $y - 2 = \frac{1}{3}(x + 4)$ and vertical transverse axis.

$$\frac{(y-2)^2}{1} - \frac{(x+4)^2}{9} = 1$$

Can you write the equation of another such hyperbola?

Any hyperbola of the form:

$$\frac{(y-2)^2}{1k} - \frac{(x+4)^2}{9k} = 1$$

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