A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY

Georgia Standards of Excellence:

MGSE9-12.F.TF.9 Prove addition, subtraction, double and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

Introduction:

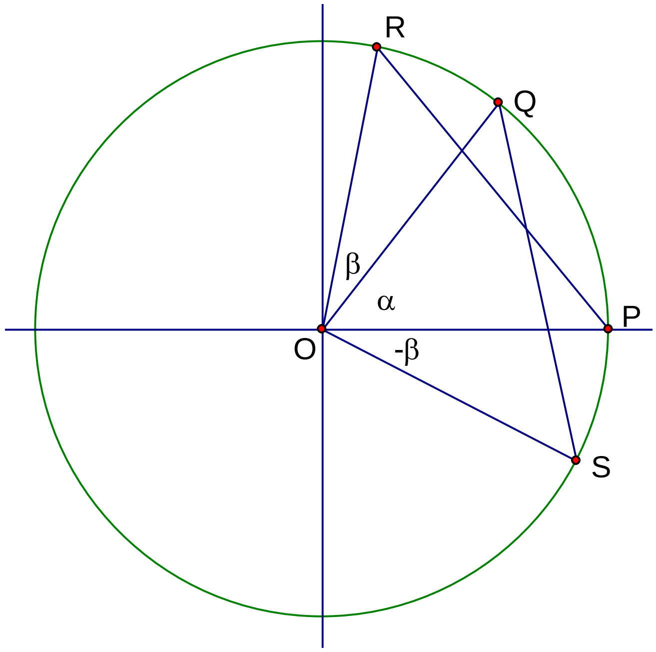
In this task, students derive the sum identity for the cosine function, in the process reviewing some of the geometric topics and ideas about proofs. This derivation also provides practice with algebraic manipulation of trigonometric functions that include examples of how applying the Pythagorean identities can often simplify a cumbersome trigonometric expression. Rewriting expressions in order to solve trigonometric equations is one of the more common applications of the sum and difference identities. This task is presented as an alternative proof to the ones the students performed earlier.

A DISTANCE FORMULA PROOF FOR THE COSINE ADDITION IDENTITY

In this task, you will use the sum and difference identities to solve equations and find the exact values of angles that are not multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Before you apply these identities to problems, you will first derive them. The first identity you will prove involves taking the cosine of the sum of two angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.



1. Complete the following congruence statements:

a. $\angle ROP \cong \angle QOS$

b. $\overline{RO} \cong \overline{QO} \cong \overline{PO} \cong \overline{SO}$

c. By the ***SAS*** congruence theorem, $\Delta ROP \cong \Delta QOS$

d. $\overline{RP} \cong \overline{QS}$

2. Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.

a. R = $(\cos(\alpha+\beta), \sin(\alpha+\beta))$

b. Q = $(\cos\alpha, \sin\alpha)$

c. P = (***1, 0***)

d. S = $(\cos(-\beta), \sin(-\beta))$

3. Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{RP}$. ***Note: Students may not simplify here, but will need to in part 5.***

Solution:

$$\sqrt{\left(\cos(\alpha+\beta)-1\right)^2+\left(\sin(\alpha+\beta)-0\right)^2}$$
$$=\sqrt{\cos^2(\alpha+\beta)-2\cos(\alpha+\beta)+1+\sin^2(\alpha+\beta)}$$ ***squaring each binomial***
$$=\sqrt{\left(\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)\right)+1-2\cos(\alpha+\beta)}$$ ***rearranging terms***
$$=\sqrt{1+1-2\cos(\alpha+\beta)}$$ ***applying a Pythagorean identity***
$$=\sqrt{2-2\cos(\alpha+\beta)}$$

4. **a.** Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{QS}$. ***Note: Students may not simplify here, but will need to in part 5.***

Solution:

$$\sqrt{\left(\cos\alpha-\cos(-\beta)\right)^2+\left(\sin\alpha-\sin(-\beta)\right)^2}$$
$$=\sqrt{\cos^2\alpha-2\cos\alpha\cos(-\beta)+\cos^2(-\beta)+\sin^2\alpha-2\sin\alpha\sin(-\beta)+\sin^2(-\beta)}$$ ***squaring each binomial***
$$=\sqrt{\left(\cos^2\alpha+\sin^2\alpha\right)+\left(\cos^2(-\beta)+\sin^2(-\beta)\right)-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$ ***rearranging terms***
$$=\sqrt{1+1-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$ ***applying a Pythagorean identity twice***
$$=\sqrt{2-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$

b. Two useful identities that you may choose to explore later are $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$. Use these two identities to simplify your solution to 4a so that your expression has no negative angles.

Solution:

$$\sqrt{2-2\cos\alpha\cos(-\beta)-2\sin\alpha\sin(-\beta)}$$
$$=\sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha(-\sin\beta)}$$
$$=\sqrt{2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta}$$

5. From 1d, you know that $\overline{RP}\cong\overline{QS}$. You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and

solve for $\cos(\alpha+\beta)$. Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for cosine.

Solution:

$\sqrt{2-2\cos(\alpha+\beta)}=\sqrt{2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta}$

$2-2\cos(\alpha+\beta)=2-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$ ***squaring both sides***

$2\cos(\alpha+\beta)=2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$ ***adding 2 to both sides***

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ***dividing both sides by 2***

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$
$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

6. Evaluate $\sin 75°$ by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30°+45°)=\sin 30°\cos 45°+\cos 30°\sin 45°$$

Solution:

$$\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{2}+\sqrt{6}}{4}$$

7. Similarly, find the exact value of the following trigonometric expressions:

a. $\cos(15°)$

Solution: $\frac{\sqrt{2}+\sqrt{6}}{4}$

b. $\sin\left(\frac{\pi}{12}\right)$

Solution: $\frac{\sqrt{6}-\sqrt{2}}{4}$

c. $\cos(345°)$

Solution: $\frac{\sqrt{2}+\sqrt{6}}{4}$

d. $\sin\frac{19\pi}{12}$

Solution: $\frac{\sqrt{2}-\sqrt{6}}{4}$

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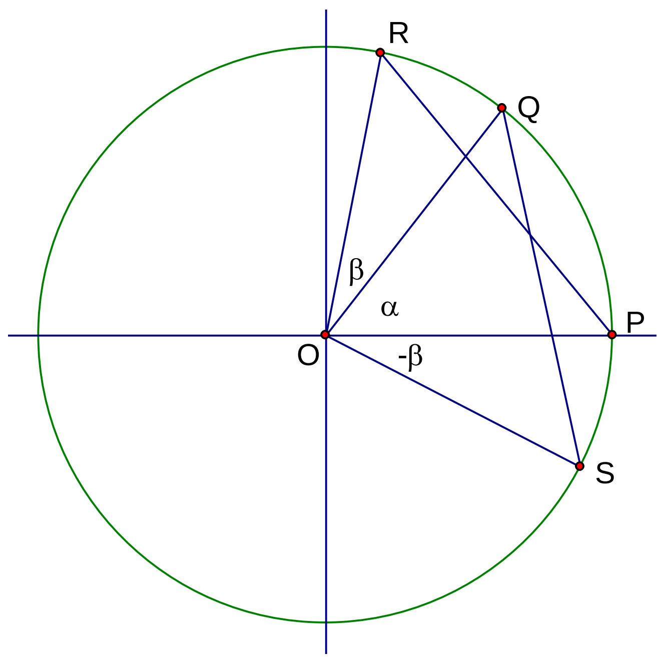
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

We can derive this identity by making deductions from the relationships between the quantities on the unit circle below.



1. Complete the following congruence statements:

a. $\angle ROP \cong$ _______

b. $\overline{RO} \cong$ ______ $\cong$ _______ $\cong$ ________

c. By the _________ congruence theorem, $\Delta ROP \cong \Delta QOS$

d. $\overline{RP} \cong$ _________

2. Write the coordinates of each of the four points on the unit circle, remembering that the cosine and sine functions produce x- and y- values on the unit circle.

a. R =

b. Q =

c. P =

d. S =

3. Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{RP}$.

4. Use the coordinates found in problem 2 and the distance formula to find the length of chord $\overline{QS}$.

5. From 1d, you know that $\overline{RP} \cong \overline{QS}$. You can therefore write an equation by setting the expressions found in problems 3 and 4b equal to one another. Simplify this equation and solve for $\cos(\alpha + \beta)$. Applying one of the Pythagorean Identities will be useful! When finished, you will have derived the angle sum identity for cosine.

The other three sum and difference identities can be derived from the identity found in problem 5. These four identities can be summarized with the following two statements.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Recall that so far, you can only calculate the exact values of the sines and cosines of multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. These identities will allow you to calculate the exact value of the sine and cosine of many more angles.

6. Evaluate $\sin 75°$ by applying the angle addition identity for sine and evaluating each trigonometric function:

$$\sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$$

7. Similarly, find the exact value of the following trigonometric expressions:

a. $\cos(15°)$

b. $\sin\left(\frac{\pi}{12}\right)$

c. $\cos(345°)$

d. $\sin\frac{19\pi}{12}$

PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES

Georgia Standards of Excellence:

MGSE9-12.F.TF.9 Prove addition, subtraction, double and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

Introduction:

This task uses algebraic manipulation and the previously developed identities to derive tangent addition and subtraction identities. Again, the emphasis should be on the mathematical processes and not the method itself. Students should feel confident in manipulating algebraic expressions and simplifying trigonometric expressions using identities.

PROVING THE TANGENT ADDITION AND SUBTRACTION IDENTITIES

By this point, you should have developed formulas for sine and cosine of sums and differences of angles. If so, you are already most of the way to finding a formula for the tangent of a sum of two angles.

Let's begin with a relationship that we already know to be true about tangent.

1. $\tan x = \dfrac{\sin x}{\cos x}$ so it stands to reason that $\tan(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)}$

Use what you already know about sum and difference formulas to expand the relationship above.

2. $\tan(x+y) = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$

3. Now we want to simplify this. (Hint: Multiply numerator and denominator by $\frac{1}{\cos x}$)

The reason to multiply by $\frac{1}{\cos x}$ ***is to establish a tangent ratio and to divide out cos x. It is important that students see why to choose that value as a part of the proof.***

$$\tan(x+y)=\frac{\frac{\sin x\cos y}{\cos x}+\frac{\cos x\sin y}{\cos x}}{\frac{\cos x\cos y}{\cos x}-\frac{\sin x\sin y}{\cos x}}=\frac{\tan x\cos y+\sin y}{\cos y-\tan x\sin y}$$

4. You can simplify it some more. Think about step 3 for a hint.

Students should multiply by $\frac{1}{\cos y}$ ***to establish a tangent ratio for the y variable and divide out cos y. Lead them back to #3 if they need help.***

$$\tan(x+y)=\frac{\frac{\tan x\cos y}{\cos y}+\frac{\sin y}{\cos y}}{\frac{\cos y}{\cos y}-\frac{\tan x\sin y}{\cos y}}=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

5. Write your formula here for the tangent of a sum:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

6. Now that you have seen the process, develop a formula for the tangent of a difference.

Encourage students to attempt this on their own without referring back to the proof of the addition identity. If they need help, they may reference it. Encourage them to persevere through this process and not give up easily. Mathematics and especially proof is meant to be a productive struggle in which students work hard to construct their own knowledge.

When they have finished, they should have: $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$

Write your formula here:

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

THE TANGENT ADDITION AND SUBTRACTION IDENTITIES

By this point, you should have developed formulas for sine and cosine of sums and differences of angles. If so, you are already most of the way to finding a formula for the tangent of a sum of two angles.

Let's begin with a relationship that we already know to be true about tangent.

1. $\tan x = \dfrac{\sin x}{\cos x}$ so it stands to reason that $\tan(x+y) =$ ________________

Use what you already know about sum and difference formulas to expand the relationship above.

2. $\tan(x+y) =$

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