DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT

Georgia Standards of Excellence:

MGSE9-12.F.TF.9 Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

Introduction:

This task guides students through a derivation of the double-angle identities for the primary trigonometric functions. It also aims to draw a distinction between doubling an angle and doubling the value of a trigonometric function associated with that angle. This distinction is a common misunderstanding that students have.

The task concentrates on the derivation of the sine double-angle formula by leading the students through a step-by-step process. The students are then left to derive the cosine and tangent identities through the use of the same process.

DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT

Before we begin…

Evaluate the following expressions without a calculator.

1a. $\cos 45° = \sqrt{2}/2$ 1b. $\cos 90° = 0$

2a. $\sin 60° = \sqrt{3}/2$ 2b. $\sin 120° = \sqrt{3}/2$

3a. $\sin \frac{\pi}{6} = 1/2$ 3b. $\sin \frac{\pi}{3} = \sqrt{3}/2$

4a. $\cos \frac{\pi}{2} = 0$ 4b. $\cos \pi = -1$

5a. $\tan \frac{5\pi}{6} = -\sqrt{3}/3$ 5b. $\tan \frac{5\pi}{3} = -\sqrt{3}$

6a. $\tan 45° = 1$ 6b. $\tan 90° =$ undefined

Study the expressions in parts a and b of the problems above – they are related. Describe how the expressions in parts a and b of the problems *differ*.

The angle measures in parts b are twice the measure of the angles in parts a.

Consider the following…

Based on the observations above decide which of these statements are 'keepers' & which are 'trash'.

If an angle is doubled then the sine value of the angle is doubled too.

If an angle is halved then the cosine value of the angle is halved too.

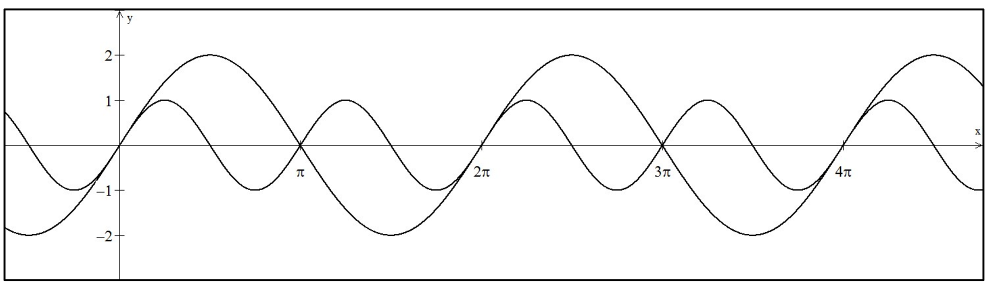
$$2\sin x \equiv \sin 2x$$

"Double the angle" & "double the sine value" really mean the same thing.

The equation $2\cos x = \cos 2x$ has *no* solutions.

Doubling angles does not double their trig values.

<u>One more thing…</u>



Before we go on, let's look at the graph above – can you identify the two equations that have been graphed? Write them on opposite sides of the "equals" sign below.

$$2\cos x = \cos 2x$$

Discussion questions: Is the equation above an identity? Does the equation above have solutions? Does the graph above change your opinion of any of the "keepers" or "trash" on the previous page?

<u>Let's Dig In!</u>

Hopefully it is obvious to you by now that if an angle is doubled, we know very little about what happens to its trig values. One thing we are certain of, however, is that *doubling an angle does not double its trig values* (except in some special cases!).

Try this: You are familiar, by now, with the identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$.
A - In the space below, rewrite the identity replacing both x and y with x.

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$

B - Now, by combining like terms on the left side and like terms on the right side, simplify the identity you wrote above.

$$\sin(2x) = 2 \sin x \cos x$$

sine cosine sum product double half identity value argument

C - Use the new mathematical identity you found to complete the following statement. The words to the right may be helpful.

To find the sine value of an angle that I've doubled, I can…

…double the product of the sine value and the cosine value

Now, on a separate paper, repeat **A - B - C -** for cosine and tangent!

DOUBLE-ANGLE IDENTITIES FOR SINE, COSINE, AND TANGENT

Before we begin…

Evaluate the following expressions without a calculator.

1a.	$\cos 45° =$	1b.	$\cos 90° =$
2a.	$\sin 60° =$	2b.	$\sin 120° =$
3a.	$\sin \frac{\pi}{6} =$	3b.	$\sin \frac{\pi}{3} =$
4a.	$\cos \frac{\pi}{2} =$	4b.	$\cos \pi =$
5a.	$\tan \frac{5\pi}{6} =$	5b.	$\tan \frac{5\pi}{3} =$
6a.	$\tan 45° =$	6b.	$\tan 90° =$

Study the expressions in parts a and b of the problems above – they are related. Describe how the expressions in parts a and b of the problems *differ*.

Consider the following…

Based on the observations above decide which of these statements are 'keepers' & which are 'trash'.

If an angle is doubled then the sine value of the angle is doubled too.

If an angle is halved then the cosine value of the angle is halved too.

$$2 \sin x \equiv \sin 2x$$

"Double the angle" & "double the sine value" really mean the same thing.

The equation $2 \cos x = \cos 2x$ has *no* solutions.

Doubling angles does not double their trig values.

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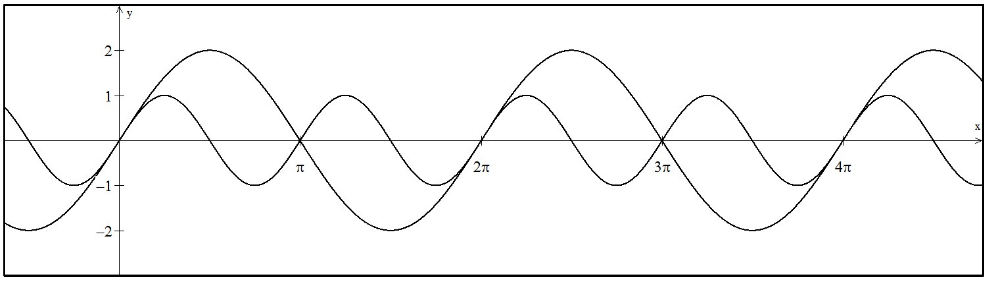

Before we go on, let's look at the graph above – can you identify the two equations that have been graphed? Write them on opposite sides of the "equals" sign below.

____________________ = ____________________

Discussion questions: Is the equation above an identity? Does the equation above have solutions? Does the graph above change your opinion of any of the "keepers" or "trash" on the previous page?

Let's Dig In!

Hopefully it is obvious to you by now that if an angle is doubled, we know very little about what happens to its trig values. One thing we are certain of, however, is that *doubling an angle does not double its trig values* (except in some special cases!).

Try this: You are familiar, by now, with the identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$.
A - In the space below, rewrite the identity replacing both *x* and *y* with *x*s.

B - Now, by combining like terms on the left side and like terms on the right side, simplify the identity you wrote above.

C - Use the new mathematical identity you found to complete the following statement. The words to the right may be helpful.
To find the sine value of an angle that I've doubled, I can…

sine
cosine
sum
product
double
half
identity
value
argument

Now, on a separate paper, repeat **A - B - C -** for cosine and tangent!

THE COSINE DOUBLE-ANGLE: A MAN WITH MANY IDENTITIES

Georgia Standards of Excellence:

MGSE9-12.F.TF.9 Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

Introduction:

This task serves as a discovery of the alternate forms of the cosine double angle identity and also as a segue into the next task (deriving half-angle identities). A discussion about the advantages of the two alternate forms is encouraged. There is an extension included that challenges students to derive the double angle identity for tangent using the double angle identity for sine and cosine.

The Cosine Double-Angle: A man with many identities.

Verify these two identities:

$$\cos^2 x - \sin^2 x = 2\cos^2 x - 1$$
$$\cos^2 x - \sin^2 x = 2\cos^2 x - (\sin^2 x + \cos^2 x)$$
$$\cos^2 x - \sin^2 x = 2\cos^2 x - \sin^2 x - \cos^2 x$$
$$\cos^2 x - \sin^2 x = 2\cos^2 x - \cos^2 x - \sin^2 x$$
$$\cos^2 x - \sin^2 x = \cos^2 x - \sin^2 x$$

$$\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$
$$\cos^2 x - \sin^2 x = (\sin^2 x + \cos^2 x) - 2\sin^2 x$$
$$\cos^2 x - \sin^2 x = \sin^2 x + \cos^2 x - 2\sin^2 x$$
$$\cos^2 x - \sin^2 x = \cos^2 x + \sin^2 x - 2\sin^2 x$$
$$\cos^2 x - \sin^2 x = \cos^2 x - \sin^2 x$$

Earlier, you discovered that $\cos(2x) = \cos^2 x - \sin^2 x$. Use the transitive property of equality along with the identities above to rewrite two alternate forms of the double angle identity for cosine.

$\cos(2x) = \cos^2 x - \sin^2 x$

or

$\cos(2x) = 2\cos^2 x - 1$

or

$\cos(2x) = 1 - 2\sin^2 x$

Discussion Question: What advantages might one of the two alternate forms of the identity have over the original?

Answers Vary: the two alternate forms express the double-angle identity for sine in terms of only one trig function instead of two.

Try this!
You have written a double-angle identity for tangent already (based off the sum identity). Try simplifying $\tan 2x = \frac{\sin 2x}{\cos 2x}$ to get the same thing. A helpful hint: You'll want to, at some point in the process, divide the top *and* bottom by… $\cos^2 x$

$$\tan 2x = \frac{\sin 2x}{\cos 2x}$$

$$\tan 2x = \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x}$$

$$\tan 2x = \frac{\frac{2\sin x \cos x}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

$$\tan 2x = \frac{\frac{2\sin x}{\cos x}}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

The Cosine Double-Angle: A Man With Many Identities

Verify these two identities:

$$\cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Earlier, you discovered that $\cos(2x) = \cos^2 x - \sin^2 x$. Use the transitive property of equality along with the identities above to rewrite two alternate forms of the double angle identity for cosine.

$\cos(2x) = \cos^2 x - \sin^2 x$

or

$\cos(2x) =$

or

$\cos(2x) =$

Discussion Question: What advantages might one of the two alternate forms of the identity have over the original?

Try this!

You have written a double-angle identity for tangent already (based off the sum identity). Try simplifying $\tan 2x = \frac{\sin 2x}{\cos 2}$ to get the same thing. A helpful hint: You'll want to, at some point in the process, divide the top *and* bottom by…

DERIVING HALF-ANGLE IDENTITIES

Georgia Standards of Excellence:

MGSE9-12.F.TF.9 Prove addition, subtraction, double, and half-angle formulas for sine, cosine, and tangent and use them to solve problems.

Introduction:

This task leads students through the derivation of the half-angle identities. It emphasizes algebraic manipulation and substitution as a means to transform a known identity into an identity that is more useful for a particular purpose. The opportunity to present math as a tool that has been created and can be manipulated by people is one that should not be missed by the teacher. It is recommended that students have completed the two tasks previous to this one.

DERIVING HALF-ANGLE IDENTITIES

After deriving the double-angle identities for sine, cosine, and tangent, you're ready to try the same for *half*-angle identities. To do so, you'll use the double-angle identities you just derived.

Follow these steps to find a half-angle identity.

1. Begin with either of the alternate forms of the cosine double-angle identity.

   Notice that, in the alternate identities, there are *two* instances of the variable x – one that is x alone and the other that is $2x$.

$$\cos 2x = 2\cos^2 x - 1 \qquad\qquad \cos 2x = 1 - 2\sin^2 x$$

2. Rearrange your chosen identity so that the term with x alone gets isolated on a side.

   Think about this question: If you wanted to evaluate angle x for the trig function you isolated, what information would you need to know in order to use the identity you have?

$$\cos 2x = 2\cos^2 x - 1 \qquad\qquad \cos 2x = 1 - 2\sin^2 x$$

$$\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}} \qquad\qquad \sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}$$

   We'd like to have something more useful that what came out of step 2. *We'd like an identity that will give us the trig function value of half an angle if we know a trig function value of the full angle.* We can do this with a simple substitution.

3. Into the result you have from step 2 above, substitute $\frac{u}{2}$ for x. Now simplify the result. Record your result below.

<u>Half-Angle Identity for Sine</u>	<u>Half-Angle Identity for Cosine</u>	<u>Half-Angle Identity for Tangent</u>
$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$	$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$	$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

4. After step 3, you have one of the three identities above. Now, return to step 1 and choose a different alternate form. Complete steps 2 and 3 again. Record the other identity in the correct place above.

5. Finally, there's tangent. It has been tricky in past identity sets (e.g. sums, differences, and double-angle) but is it surprisingly easy to come up with a working identity for $\tan\frac{u}{2}$.

Verify this identity (there are three here) and discuss your findings with a classmate:

$$\frac{\sin x}{1+\cos x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x}$$

This triple identity has three parts. The middle member of this identity is the half angle identity for tangent that was derived above. The two members on either side of the identity above are alternate forms of the half angle identity for tangent. Students could work through any one of these three identities above but should not miss the fact that *alternate forms of the half angle identity for tangent are being verified*.