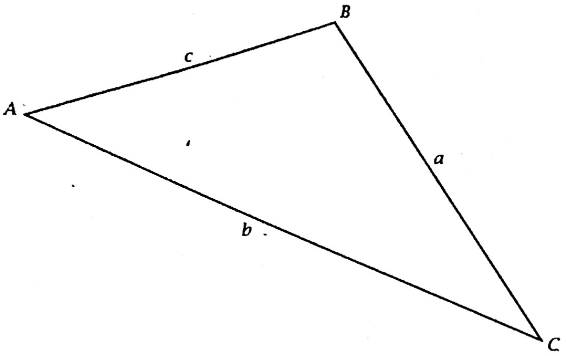
Name: ______________________ Group Members: ______________________

Exploration 6-2b: Angles by Law of Cosines

Date: __________

Objective: Use the law of cosines to calculate the measure of an angle of a triangle if three sides are known.



1. For $\triangle ABC$, write the law of cosines in the form involving cos *B*.

   $b^2 = a^2 + c^2 - 2ac \cdot \cos B$

2. Measure the sides of $\triangle ABC$ to the nearest 0.1 cm.

   $a \approx$ 5.3 $b \approx$ 7.5 $c \approx$ 4.3

3. Substitute the values of *a*, *b*, and *c* from Problem 2 into the law of cosines from Problem 1. Solve the resulting equation for *B*. Store the answer without rounding.

   $b^2 = a^2 + c^2 - 2ac \cdot \cos B$

   $B = \cos^{-1}\left(\frac{b^2 - a^2 - c^2}{-2ac}\right) = \left(\frac{a^2+c^2-b^2}{2ac}\right)$

   $B = \cos^{-1}\left(\frac{5.3^2 + 4.3^2 - 7.5^2}{2(5.3)(4.3)}\right)$

   $B = 102^\circ$

   $b^2 + 2ac \cdot \cos B = a^2 + c^2$

   $2ac \cdot \cos B = a^2 + c^2 - b^2$

   $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

   $B = \cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right)$

4. Measure angle *B* with a protractor. $\approx 103^\circ$

5. How close did your calculated angle value come to the measured value?

6. Without substituting values for *a*, *b*, and *c*, solve the equation from Problem 1 for cos *B* in terms of *a*, *b*, and *c*. Simplify as much as possible.

   $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

7. By observing the pattern in your answer to Problem 6, write an equation for cos *A* in terms of *a*, *b*, and *c*.

   $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

8. Use your answer to Problem 7 and the measured values of *a*, *b*, and *c* to calculate the measure of angle *A*. Store the answer without rounding.

   $A = \cos^{-1}\left(\frac{7.5^2 + 4.3^2 - 5.3^2}{2(7.5)(4.3)}\right) \approx 44^\circ$

9. Use the pattern in Problem 6 to calculate the measure of angle *C* by the law of cosines. Store the answer without rounding.

   $C = \cos^{-1}\left(\frac{5.3^2 + 7.5^2 - 4.3^2}{2(5.3)(7.5)}\right)$

   $C \approx 34^\circ$

   102°
   44°
   34°
   180°

10. Use the stored values of *A*, *B*, and *C* to show that the sum of the angles in the triangle is 180°.

    $A + B + C = 180^\circ$!!

11. Measure angles *A* and *C*.

    $A \approx$ 44° $C \approx$ 34

12. How close do your calculated angle values come to the measured values?

    ✓

13. What did you learn as a result of doing this Exploration that you did not know before?

$\cos A = \left(\dfrac{b^2+c^2-a^2}{2ac}\right)$

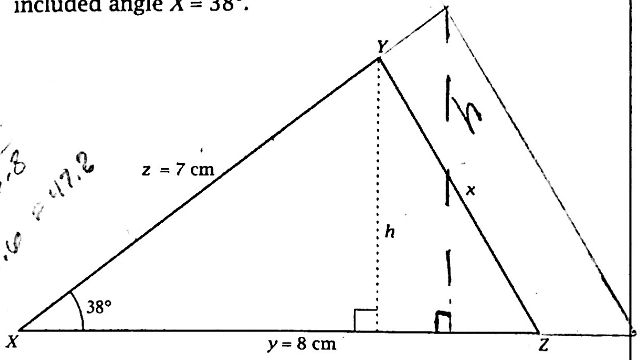
Name: ______________________ Group Members: ______________________

Exploration 6-3a: Area of a Triangle and Hero's Formula

Date: ____________

Objective: Derive a *quick* method to calculate the area of a triangle from two sides and the included angle.

For Problems 1-3, $\triangle XYZ$ has sides $y = 8$ cm, $z = 7$ cm, and included angle $X = 38°$.



$\sin X = \frac{h}{z}$

$h = z \sin X$

1. Do you agree with the given measurements?

   y ______ z ______ $\angle X$ 38°

2. Use the given measurements to calculate altitude h. Measure h. Does it agree with the calculation?

   $\sin 38 = \frac{h}{7}$

   $h = 7 \cdot \sin 38$

   $h = 4.31$ cm

   h = 3.6 por measuring

3. You recall from geometry that the area of a triangle is $\frac{1}{2}$(base)(altitude). Find the area of $\triangle XYZ$.

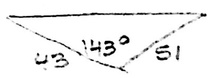
   $A = \frac{1}{2}(8)(4.31) = 17.24 \text{ cm}^2$

4. By substituting $z \sin X$ in Problem 3 you get

   Area $= \frac{1}{2}y\underbrace{z \sin X}_{h}$, or, in general,

   Area $= \frac{1}{2}$(side)(side)(sine of included angle).

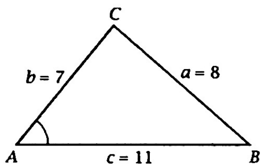
   Sketch a triangle with sides 43 m and 51 m, and included angle 143°. Use this area formula to find the area of this triangle.



   $A = \frac{1}{2}(43)(51)\sin(143)$

   $A = 659.39 \text{ m}^2$

For Problems 5-8, $\triangle ABC$ has sides $a = 8$, $b = 7$, and $c = 11$.



5. Find the measure of angle A using the law of cosines. Store the answer without rounding.

   $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

   $A = \cos^{-1}\left(\frac{a^2 - b^2 - c^2}{-2bc}\right) = \cos^{-1}\left(\frac{8^2 - 7^2 - 11^2}{-2(7)(11)}\right)$

   $A = 46.5°$

6. Use the unrounded value of A and the area formula of Problem 3 to find the area of $\triangle ABC$.

   $\text{Area} = \frac{1}{2}(11)\underbrace{(7)(\sin A)}_{ht}$

   $= 27.93 \text{ u}^2$

7. Calculate the **semiperimeter** (half the perimeter) of the triangle, $s = \frac{1}{2}(a + b + c)$.

   $s = \frac{1}{2}(7+8+11) = 13 \text{ u}$

8. Evaluate the quantity $\sqrt{s(s-a)(s-b)(s-c)}$. What interesting thing do you notice about the answer?

   $\sqrt{13(13-8)(13-7)(13-11)}$

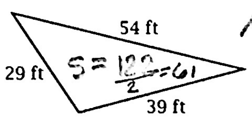
   $= \sqrt{13(5)(6)(2)} = \sqrt{780} = 27.93 \text{ u}^2$

9. Use **Hero's formula**, namely,

   Area $= \sqrt{s(s-a)(s-b)(s-c)}$

   to find the area of this triangle.

   $s = \frac{29+54+39}{2} = 61$



   $\text{Area} = \sqrt{61(61-29)(61-54)(61-39)}$

   $= \sqrt{300,608}$

   $= 548.28 \text{ ft}^2$

10. What did you learn as a result of doing this Exploration that you did not know before?

x^2-4

$(x+2)(x-2)$

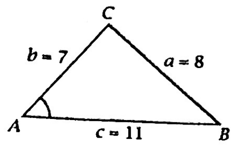
Name: ______________________ Group Members: ______________________

Exploration 6-3b: Derivation of Hero's Formula

Date: __________

Objective: Derive Hero's formula for calculating the area of a triangle given the measures of the three sides.

The figure shows $\triangle ABC$ with sides $a = 8$, $b = 7$, and $c = 11$.



1. Calculate the area of $\triangle ABC$ again using **Hero's formula:**

   $$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

   where s is the **semiperimeter** of the triangle (half the perimeter). $s = \frac{8+7+11}{2} = \frac{26}{2} = 13$

   $\text{Area} = \sqrt{13(5)(6)(2)} = 780$

   $\text{Area} = 27.93\,u^2$

2. The area formula from Section 6-3 is

   $$\text{Area} = \frac{1}{2}bc\sin A$$

   Explain why this can be written as

   $$\text{Area} = \sqrt{\frac{1}{4}b^2c^2(1-\cos^2 A)}$$

   $\sin^2 A + \cos^2 A = 1 \Rightarrow \sin^2 A = 1 - \cos^2 A$

   $\sin A = \sqrt{1-\cos^2 A}$

   $(\frac{1}{2}bc)^2 = \frac{1}{4}b^2c^2 \quad \therefore \frac{1}{2}bc = \sqrt{\frac{1}{4}b^2c^2}$

   $\text{Area} = \sqrt{}\cdot\sqrt{} = \sqrt{\frac{1}{4}b^2c^2(1-\cos^2 A)}$

3. Explain why the result of Problem 2 can be written as

   $$\text{Area} = \sqrt{\frac{1}{2}bc(1+\cos A)\cdot\frac{1}{2}bc(1-\cos A)}$$

   DOTS: $1-\cos^2 A = (1-\cos A)(1+\cos A)$

   $\text{Area} = \sqrt{\frac{1}{4}b^2c^2(1+\cos A)(1-\cos A)}$

   $\text{Area} = \sqrt{\underbrace{\frac{1}{2}bc(1+\cos A)}_{\#5}\cdot\underbrace{\frac{1}{2}bc(1-\cos A)}_{\#6}}$

4. Use the law of cosines to write $\cos A$ in terms of a, b, and c *without* substituting the values.

   $a^2 = b^2 + c^2 - 2bc\cdot\cos A$

   $\cos A = \frac{a^2-b^2-c^2}{-2bc} = \frac{(b^2+c^2)-a^2}{2bc}$

5. Show that $\frac{1}{2}bc(1+\cos A)$ can be written as

   $$\frac{(b+c)^2-a^2}{4} = \frac{b+c+a}{2}\cdot\frac{b+c-a}{2}$$

   $\frac{1}{2}bc(1+\cos A)$

   $= \frac{1}{2}bc\left(1+\frac{(b^2+c^2)-a^2}{2bc}\right)$

   $= \frac{1}{2}bc + \frac{bc}{2}\left(\frac{(b^2+c^2)-a^2}{2bc}\right)$ bc cancels

   $= \frac{bc}{2} + \frac{(b^2+c^2)-a^2}{4}$ common den.

   $= \frac{2bc}{4} + \frac{b^2+c^2}{4} - \frac{a^2}{4}$ group b's & c's

   $= \frac{b^2+2bc+c^2}{4} - \frac{a^2}{4}$ Perfect Sq Trinomial

   $= \frac{(b+c)^2}{4} - \frac{a^2}{4}$ $\Big\}$ $\left(\frac{b+c}{2}+\frac{a}{2}\right)\left(\frac{b+c}{2}-\frac{a}{2}\right)$

   $= \left(\frac{b+c+a}{2}\right)\left(\frac{b+c-a}{2}\right)$ Dots

(Over)

6. By substitutions such as you did in Problem 5, it is possible to transform the other factor under the radical in Problem 3 this way. (It is not necessary for you to *do* the transformations.)

$$\frac{1}{2}bc(1 - \cos A) = \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}$$

Using this information and the result of Problem 5, the area can be written as

$$\text{Area} = \sqrt{\frac{b+c+a}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}}$$

The first fraction under the radical is the semiperimeter, $s = \frac{1}{2}(a+b+c)$. Show that the other three fractions can be written as $(s-a)$, $(s-b)$, and $(s-c)$, respectively. From the results, derive Hero's formula,

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Area} = \sqrt{\tfrac{1}{2}bc(1+\cos A) \cdot \tfrac{1}{2}bc(1-\cos A)}$$

$$= \sqrt{\underbrace{\frac{b+c-a}{2}}_{\frac{2s-a-a}{2} = \frac{2s-2a}{2}} \cdot \underbrace{\frac{b+c+a}{2}}_{s} \cdot \underbrace{\frac{a-b+c}{2}}_{\frac{a+c-b}{2} = \frac{2s-b-b}{2}} \cdot \underbrace{\frac{a+b-c}{2}}_{\frac{2s-c-c}{2}}}$$

$(s-a)$ (s) $(s-b)$ $(s-c)$

$s = \frac{1}{2}(a+b+c)$

$2s = a+b+c$

$b+c = 2s-a$

OR

$a+c = 2s-b$

OR

$a+b = 2s-c$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

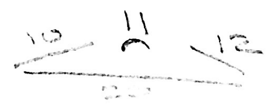
7. Use Hero's formula to find the area of ΔXYZ if $x = 50$ cm, $y = 60$ cm, and $z = 80$ cm.

$s = 0.5(50+60+80) = 95$

$\text{Area} = \sqrt{95(95-50)(95-60)(95-80)}$ $\sqrt{95(45)(35)(15)}$

$= 1498.12 \text{ cm}^2$

8. Show that Hero's formula indicates that there is no possible ΔBAD if $b = 10$ ft, $a = 12$ ft, and $d = 26$ ft.



$s = 0.5(10+12+26) = 24$

$\text{Area} = \sqrt{24(24-10)(24-12)(24-26)} = 24(14)(12)(-2)$

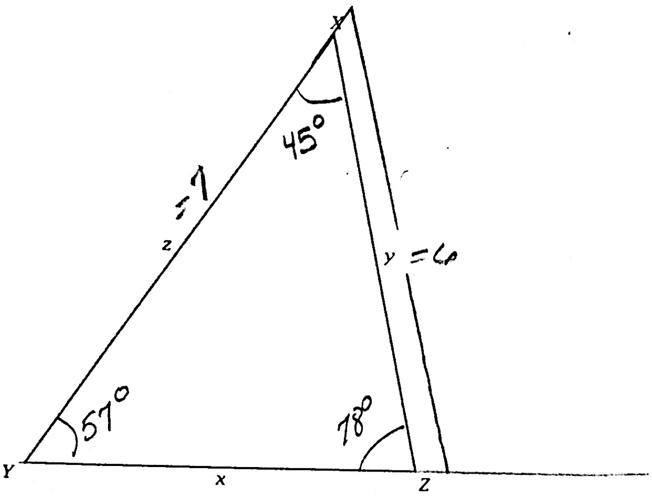
9. What did you learn as a result of doing this Exploration that you did not know before?

Name: ______________ Group Members: ______________

Exploration 6-4a: The Law of Sines

Date: ______

Objective: Use the ratio of a side length to the sine of the opposite angle to find other parts of a triangle.



1. In $\triangle XYZ$, are the following measurements correct?

   $y = 6.0$ cm ≈ 5.7 $z = 7.0$ cm ≈ 6.6

   $Y = 57°$ ______ $Z = 78°$ ______

2. Assuming that the measurements in Problem 1 are correct, calculate these ratios:

   $\dfrac{y}{\sin Y} = \dfrac{6}{\sin 57} = 7.15$

   $\dfrac{z}{\sin Z} = \dfrac{7}{\sin 78} = 7.15$

3. The **law of sines** states that within a triangle, the ratio of the length of a side to the sine of the opposite angle is constant. Do the calculations in Problem 2 seem to confirm this property? ☺

4. Measure angle X. 45°

5. Assuming that the law of sines is correct,

   $$\frac{x}{\sin X} = \frac{y}{\sin Y}$$

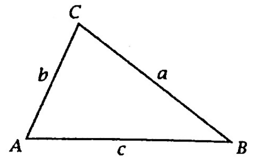
   Use this information and the measured value of X to calculate the length x.

   $$\frac{x}{\sin 45} = \frac{6}{\sin 57}$$

   $$x = \frac{6 \cdot \sin 45}{\sin 57} = 5.06 \text{ cm}$$

6. Measure side x. Does your measurement agree with the calculated value in Problem 5? 5.1 ☺

The law of sines can be derived **algebraically.**



7. For $\triangle ABC$, use the area formula to write the area *three* ways: one involving angle A, one involving angle B, and one involving angle C.

   Area $= \frac{1}{2} bc \cdot \sin A$

   " $= \frac{1}{2} ac \cdot \sin B$

   " $= \frac{1}{2} ab = \sin C$

8. The area of a triangle is *independent* of the way you *measure* that area, so all three area expressions in Problem 7 are equal to each other. Write a *three*-part equation expressing this fact.

   $$\frac{1}{2} bc \cdot \sin A = \frac{1}{2} ac \cdot \sin B = \frac{1}{2} ab \cdot \sin C$$

9. Divide all three "sides" of the equation in Problem 8 by whatever is necessary to leave only the sines of the angles in the numerators. Simplify.

   $$\frac{2}{abc} \cdot [\quad]$$

   $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

10. The equation you should have gotten in Problem 9 is the **law of sines.** Explain why it is equivalent to the law of sines as written in Problem 5.

11. What did you learn as a result of doing this Exploration that you did not know before?

0.7806

$\frac{65}{100}$

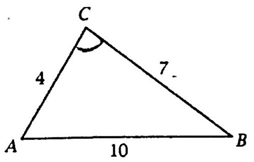
Name: ______________________ Group Members: ______________________

Exploration 6-4b: The Law of Sines for Angles

Date: __________

Objective: Discover the hazards of using the law of sines to find an angle of a triangle.

The figure shows $\triangle ABC$ (not to scale) with $a = 7$, $b = 4$, and $c = 10$.



1. Use the law of cosines to find the measure of angle A. Store the answer in your calculator without round-off.

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$A = \cos^{-1}\left(\frac{a^2 - b^2 - c^2}{-2bc}\right) = \frac{b^2 + c^2 - a^2}{2bc}$$

$$A = \cos^{-1}\left(\frac{7^2 - 4^2 - 10^2}{-2(4)(10)}\right)$$

$$= 33.12^\circ$$

(33.12294021)

2. Use the answer to Problem 1 and the law of sines to calculate the measure of angle C.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad a \cdot \sin C = c \cdot \sin A$$

$$C = \sin^{-1}\left(\frac{c \cdot \sin A}{a}\right) = \sin^{-1}\left(\frac{10 \cdot \sin 33.13}{7}\right)$$

$$C = 51.32^\circ$$

3. Calculate the measure of angle C again, directly from the three side lengths, using the law of cosines.

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

$$C = \cos^{-1}\left(\frac{c^2 - a^2 - b^2}{-2ab}\right)$$

$$C = \cos^{-1}\left(\frac{10^2 - 7^2 - 4^2}{-2(7)(4)}\right)$$

$$C = 128.68^\circ$$

$$\cos C = \frac{c^2 - a^2 - b^2}{-2ab} = \frac{a^2 + b^2 - c^2}{2ab}$$

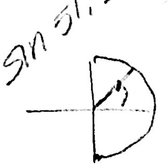
$$C = \cos^{-1}\left(\frac{7^2 + 4^2 - 10^2}{2(7)(4)}\right) = 128.68^\circ$$

$\frac{51.32}{90}$ 38.68

4. Do your answers to Problems 2 and 3 agree? If not, describe how you can use the *general* solution for inverse sine (arcsin instead of $\sin^{-1}$) to get the correct answer by the law of sines.

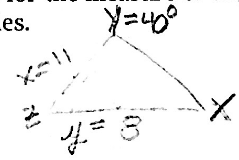
$C\ 51.3^\circ$

$\sin 51.3 = 0.7804$



$$\begin{array}{r} 180.00 \\ -51.32 \\ \hline 128.68 \end{array}$$

5. Triangle XYZ has $x = 11$ in., $y = 8$ in., and angle $Y = 40^\circ$. Use the law of sines to find *two* possible values for the measure of angle X. Sketch both triangles.



$$\frac{11}{\sin X} = \frac{8}{\sin 40}$$

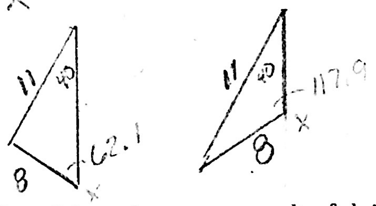
$$8 \sin X = 11 \sin 40$$

$$\sin X = \frac{11 \sin 40}{8}$$

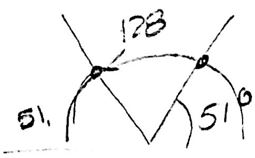
$$X = \sin^{-1}\left(\frac{11 \sin 40}{8}\right) = 62.1^\circ$$

OR

$$X = 117.9^\circ$$



6. What did you learn as a result of doing this Exploration that you did not know before?

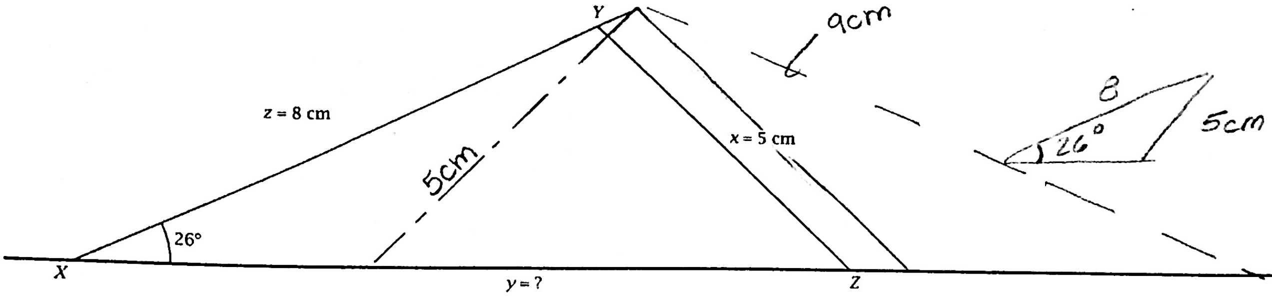


Name: ______________________ Group Members: ______________________

Exploration 6-5a: The Ambiguous Case, SSA

Date: __________

Objective: Investigate what can happen if SSA (side, side, angle) is given in a triangle.



1. In $\triangle XYZ$, side $x = 5$ cm, side $z = 8$ cm, and angle $X = 26°$. Do you agree with these measurements?

   ____

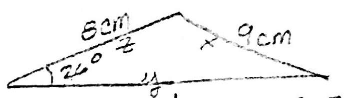
2. Draw another possibility for $\triangle XYZ$ with the same values of x, z, and X but a different value of y. For both triangles, measure side y.

   $y \approx$ 10.8 cm or $y \approx$ 3.6 cm

3. What does the word **ambiguous** mean? Why is case SSA called *ambiguous*?

   - two or more answers
   - there are 2 answers

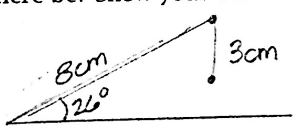
4. If side z and angle X remain fixed and side x is increased to 9 cm, will there still be two possible triangles? Draw the result on the given figure.

   Only one answer

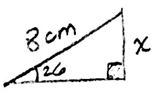
   ≈ 15.5 cm

5. If side z and angle X remain fixed and side x is decreased to 3 cm, how many possible triangles will there be? Show your conclusion on the given figure.

   • No $\triangle$, too short

6. When side x is perpendicular to side y, there is exactly one triangle. Calculate the value of x in this case. Confirm that it is correct by taking its measurement on the figure.

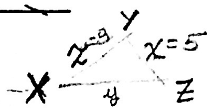
   

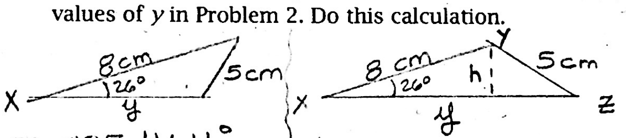
   $\sin 26 = \frac{x}{8}$

   $x = 8\sin 26$

   $x = 3.51$

7. By clever application of the law of cosines and the quadratic formula, it is possible to *calculate* both values of y in Problem 2. Do this calculation.

   $Z = 135.4611°$

   $Y = 18.5389$

   $x^2 = y^2 + z^2 - 2yz\cos X$

   $5^2 = y^2 + 8^2 - 2y8\cos 26$

   $y^2 - 16y\cos 26 + 64 - 25 = 0$

   $y^2 - 16\cos 26(y) + 39 = 0$

   $\frac{+16\cos 26 \pm \sqrt{(16\cos 26)^2 - 4(1)(39)}}{2(1)}$

   $\frac{14.3807 \pm 7.12774}{2}$

   10.75 or 3.62 cm

8. Show how the technique of Problem 7 justifies the answer to Problem 4.

   $9^2 = 8^2 + y^2 - 2 \cdot 8 \cdot y \cos 26$

   $81 = 64 + y^2 - 16\cos 26 \cdot y$

   $y^2 - 16\cos 26(y) - 17 = 0$

   $y = \frac{16\cos 26 \pm \sqrt{(16\cos 26)^2 - 4(1)(-17)}}{2(1)} = \frac{14.3807 \pm 16.5772}{2}$

   $= 15.48$ cm

9. Show how the technique of Problem 7 justifies the answer to Problem 5.

   $3^2 = 8^2 + y^2 - 2 \cdot 8 \cdot y \cos 26$

   $9 = 64 + y^2 - 16\cos 26(y)$

   $0 = y^2 - 16\cos 26(y) + 55$

   $y = \frac{16\cos 26 \pm \sqrt{(-16\cos 26)^2 - 4(1)(55)}}{2} =$

   $\frac{16\cos 26 \pm \sqrt{-13}}{2}$

   $= \frown$

10. What did you learn as a result of doing this Exploration that you did not know before?