14.7 Using Double- and Half-Angle Formulas

What you should learn

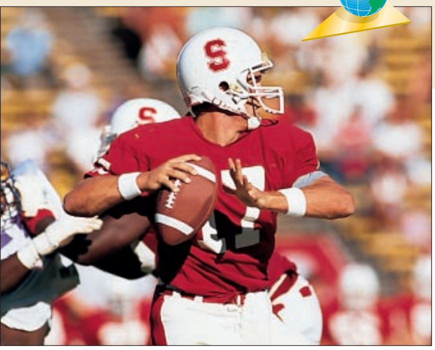
GOAL 1 Evaluate expressions using double- and half-angle formulas.

GOAL 2 Use double- and half-angle formulas to solve **real-life** problems, such as finding the mach number for an airplane in **Ex. 70**.

Why you should learn it

▼ To model **real-life** situations with double- and half-angle relationships, such as kicking a football in **Example 8**.

REAL LIFE



GOAL 1 DOUBLE- AND HALF-ANGLE FORMULAS

In this lesson you will use formulas for double angles (angles of measure $2u$) and half angles $\left(\text{angles of measure } \frac{u}{2}\right)$. The three formulas for $\cos 2u$ below are equivalent, as are the two formulas for $\tan \frac{u}{2}$. Use whichever formula is most convenient for solving a problem.

DOUBLE-ANGLE AND HALF-ANGLE FORMULAS

DOUBLE-ANGLE FORMULAS

$\cos 2u = \cos^2 u - \sin^2 u$ $\qquad$ $\sin 2u = 2 \sin u \cos u$

$\cos 2u = 2\cos^2 u - 1$ $\qquad$ $\tan 2u = \dfrac{2 \tan u}{1 - \tan^2 u}$

$\cos 2u = 1 - 2\sin^2 u$

HALF-ANGLE FORMULAS

$\sin \frac{u}{2} = \pm\sqrt{\dfrac{1 - \cos u}{2}}$ $\qquad$ $\tan \frac{u}{2} = \dfrac{1 - \cos u}{\sin u}$

$\cos \frac{u}{2} = \pm\sqrt{\dfrac{1 + \cos u}{2}}$ $\qquad$ $\tan \frac{u}{2} = \dfrac{\sin u}{1 + \cos u}$

The signs of $\sin \frac{u}{2}$ and $\cos \frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

EXAMPLE 1 *Evaluating Trigonometric Expressions*

Find the exact value of **(a)** $\tan \frac{\pi}{8}$ and **(b)** $\cos 105°$.

SOLUTION

a. Use the fact that $\frac{\pi}{8}$ is half of $\frac{\pi}{4}$.

$$\tan \frac{\pi}{8} = \tan \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{1 - \cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$

b. Use the fact that 105° is half of 210° and that cosine is negative in Quadrant II.

$$\cos 105° = \cos \frac{1}{2}(210°) = -\sqrt{\frac{1 + \cos 210°}{2}}$$

$$= -\sqrt{\frac{1 + \left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{4}} = -\frac{\sqrt{2 - \sqrt{3}}}{2}$$

STUDENT HELP

Study Tip

In Example 1 note that, in general, $\tan \frac{u}{2} \neq \frac{1}{2} \tan u$. Similar statements can be made for the other trigonometric functions of double and half angles.

> **STUDENT HELP**
>
> **Study Tip**
> Because $\pi < u < \frac{3\pi}{2}$ in Example 2, you can multiply through the inequality by $\frac{1}{2}$ to get $\frac{\pi}{2} < \frac{u}{2} < \frac{3\pi}{4}$, so $\frac{u}{2}$ is in Quadrant II.

EXAMPLE 2 *Evaluating Trigonometric Expressions*

Given $\cos u = -\frac{3}{5}$ with $\pi < u < \frac{3\pi}{2}$, find the following.

a. $\sin 2u$ **b.** $\sin \frac{u}{2}$

SOLUTION

a. Use a Pythagorean identity to conclude that $\sin u = -\frac{4}{5}$.

$$\sin 2u = 2 \sin u \cos u$$
$$= 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{24}{25}$$

b. Because $\frac{u}{2}$ is in Quadrant II, $\sin \frac{u}{2}$ is positive.

$$\sin \frac{u}{2} = \sqrt{\frac{1 - \cos u}{2}} = \sqrt{\frac{1 - \left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$$

EXAMPLE 3 *Simplifying a Trigonometric Expression*

Simplify $\frac{\cos 2\theta}{\sin \theta + \cos \theta}$.

> **STUDENT HELP**
>
> **Study Tip**
> Because there are three formulas for cos 2*u*, you will want to choose the one that allows you to simplify the expression in which cos 2*u* appears, as illustrated in Example 3.

SOLUTION

$$\frac{\cos 2\theta}{\sin \theta + \cos \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta + \cos \theta}$$ **Use a double-angle formula.**

$$= \frac{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)}{\sin \theta + \cos \theta}$$ **Factor difference of squares.**

$$= \cos \theta - \sin \theta$$ **Simplify.**

EXAMPLE 4 *Verifying a Trigonometric Identity*

Verify the identity $\sin 3x = 3 \sin x - 4 \sin^3 x$.

SOLUTION

$$\sin 3x = \sin (2x + x)$$ **Rewrite sin 3*x* as sin (2*x* + *x*).**

$$= \sin 2x \cos x + \cos 2x \sin x$$ **Use a sum formula.**

$$= (2 \sin x \cos x) \cos x + (1 - 2 \sin^2 x) \sin x$$ **Use double-angle formulas.**

$$= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x$$ **Multiply.**

$$= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x$$ **Use a Pythagorean identity.**

$$= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x$$ **Distributive property**

$$= 3 \sin x - 4 \sin^3 x$$ **Combine like terms.**

EXAMPLE 5 *Solving a Trigonometric Equation*

STUDENT HELP

INTERNET HOMEWORK HELP
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Solve $\tan 2x + \tan x = 0$ for $0 \le x < 2\pi$.

SOLUTION

$$\tan 2x + \tan x = 0 \quad \text{Write original equation.}$$

$$\frac{2\tan x}{1 - \tan^2 x} + \tan x = 0 \quad \text{Use a double-angle formula.}$$

$$2\tan x + \tan x\,(1 - \tan^2 x) = 0 \quad \text{Multiply each side by } 1 - \tan^2 x.$$

$$2\tan x + \tan x - \tan^3 x = 0 \quad \text{Distributive property.}$$

$$3\tan x - \tan^3 x = 0 \quad \text{Combine like terms.}$$

$$\tan x\,(3 - \tan^2 x) = 0 \quad \text{Factor.}$$

Set each factor equal to 0 and solve for x.

$$\tan x = 0 \qquad \text{or} \qquad 3 - \tan^2 x = 0$$

$$x = 0, \pi \qquad\qquad 3 = \tan^2 x$$

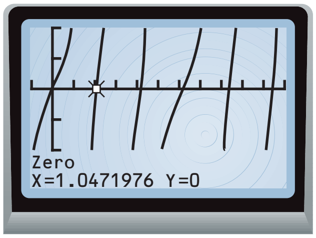
$$\pm\sqrt{3} = \tan x$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

✓**CHECK** You can use a graphing calculator to check the solutions. Graph the following function:

$$y = \tan 2x + \tan x$$

Then use the *Zero* feature to find the x-values for which $y = 0$.



. . . .

Some equations that involve double or half angles can be solved directly—without resorting to double- or half-angle formulas.

EXAMPLE 6 *Solving a Trigonometric Equation*

Solve $2\cos\frac{x}{2} + 1 = 0$.

SOLUTION

$$2\cos\frac{x}{2} + 1 = 0 \quad \text{Write original equation.}$$

$$2\cos\frac{x}{2} = -1 \quad \text{Subtract 1 from each side.}$$

$$\cos\frac{x}{2} = -\frac{1}{2} \quad \text{Divide each side by 2.}$$

$$\frac{x}{2} = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad \frac{4\pi}{3} + 2n\pi \quad \text{General solution for } \frac{x}{2}$$

$$x = \frac{4\pi}{3} + 4n\pi \quad \text{or} \quad \frac{8\pi}{3} + 4n\pi \quad \text{General solution for } x$$

GOAL 2 USING TRIGONOMETRY IN REAL LIFE

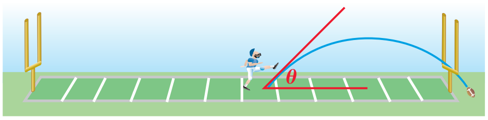
The path traveled by an object that is projected at an initial height of h_0 feet, an initial speed of v feet per second, and an initial angle θ is given by

$$y = -\frac{16}{v^2 \cos^2 \theta}x^2 + (\tan \theta)x + h_0$$

where x and y are measured in feet. (This model neglects air resistance.)

EXAMPLE 7 *Simplifying a Trigonometric Model*

SPORTS Find the horizontal distance traveled by a football kicked from ground level ($h_0 = 0$) at speed v and angle θ.



Not drawn to scale

SOLUTION

Using the model above with $h_0 = 0$, set y equal to 0 and solve for x.

$$-\frac{16}{v^2 \cos^2 \theta}x^2 + (\tan \theta)x = 0$$ **Let $y = 0$.**

$$(-x)\left(\frac{16}{v^2 \cos^2 \theta}x - \tan \theta\right) = 0$$ **Factor.**

$$\frac{16}{v^2 \cos^2 \theta}x - \tan \theta = 0$$ **Zero product property (Ignore $-x = 0$.)**

$$\frac{16}{v^2 \cos^2 \theta}x = \tan \theta$$ **Add $\tan \theta$ to each side.**

$$x = \frac{1}{16}v^2 \cos^2 \theta \tan \theta$$ **Multiply each side by $\frac{1}{16}v^2 \cos^2 \theta$.**

$$x = \frac{1}{16}v^2 \cos \theta \sin \theta$$ **Use $\cos \theta \tan \theta = \sin \theta$.**

$$x = \frac{1}{32}v^2 (2 \cos \theta \sin \theta)$$ **Rewrite $\frac{1}{16}$ as $\frac{1}{32} \cdot 2$.**

$$x = \frac{1}{32}v^2 \sin 2\theta$$ **Use a double-angle formula.**

FOCUS ON APPLICATIONS

REAL LIFE **FIELD GOALS** The longest professional field goal was 63 yards, made by Tom Dempsy in 1970. This record was tied by Jason Elam during the 1998–1999 season.

INTERNET APPLICATION LINK www.mcdougallittell.com

EXAMPLE 8 *Using a Trigonometric Model*

SPORTS You are kicking a football from ground level with an initial speed of 80 feet per second. Can you make the ball travel 200 feet?

SOLUTION

$$200 = \frac{1}{32}(80)^2 \sin 2\theta$$ **Substitute for x and v in the formula from Example 7.**

$$1 = \sin 2\theta$$ **Divide each side by $\frac{1}{32}(80)^2 = 200$.**

$$90° = 2\theta$$ **$\sin^{-1} 1 = 90°$**

$$45° = \theta$$ **Solve for θ.**

▶ You can make the football travel 200 feet if you kick it at an angle of 45°.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: $\sin 2u = 2 \sin u \cos u$ is called the __?__ formula for sine.

Concept Check ✓

2. Suppose you want to simplify $\frac{\cos 2\theta - \cos \theta}{\cos \theta - 1}$. Which double-angle formula for cosine would you use to rewrite $\cos 2\theta$? Explain.

3. ERROR ANALYSIS Explain what is wrong in the calculations shown below.

a.

$$\tan 15° = \tan \frac{30°}{2} = \frac{1}{2} \tan 30° = \frac{1}{2}\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{6}$$

b.

$$\cos 195° = \cos \frac{390°}{2} = \sqrt{\frac{1 - \sin 390°}{2}} = \sqrt{\frac{1 - \frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Skill Check ✓

Given $\tan u = \frac{2}{5}$ with $\pi < u < \frac{3\pi}{2}$, find the exact value of the expression.

4. $\sin \frac{u}{2}$ **5.** $\cos \frac{u}{2}$ –   **6.** $\tan \frac{u}{2}$

7. $\sin 2u$ **8.** $\cos 2u$ **9.** $\tan 2u$

Simplify the expression.

10. $\cos^2 x - 2 \cos 2x + 1$ **11.** $\sin 2x \tan \frac{x}{2}$ **12.** $\sin^2 x + \sin 2x + \cos^2 x$

13. $\frac{\tan 2x}{\sec^2 x}$ **14.** $\sin \frac{x}{2} \cos \frac{x}{2}$ **15.** $\cos 2x \cot^2 x - \cot^2 x$

16. **FOOTBALL** Look back at Example 8 on page 878. Through what range of angles can you kick the football to make it travel at least 150 feet?

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 960.

EVALUATING TRIGONOMETRIC EXPRESSIONS **Find the exact value of the expression.**

17. $\tan 15°$ **18.** $\sin 22.5°$ **19.** $\tan (-22.5°)$

20. $\cos 67.5°$ **21.** $\tan (-75°)$ **22.** $\cos \frac{7\pi}{8}$

23. $\sin -\frac{7\pi}{12}$ **24.** $\cos -\frac{5\pi}{12}$ **25.** $\sin \frac{\pi}{12}$

HALF-ANGLE FORMULAS **Find the exact values of $\sin \frac{u}{2}$, $\cos \frac{u}{2}$, and $\tan \frac{u}{2}$.**

26. $\cos u = \frac{3}{5}, 0 < u < \frac{\pi}{2}$ **27.** $\cos u = \frac{2}{3}, \frac{3\pi}{2} < u < 2\pi$

28. $\sin u = \frac{9}{10}, \frac{\pi}{2} < u < \pi$ **29.** $\sin u = -\frac{4}{5}, \frac{3\pi}{2} < u < 2\pi$

> **STUDENT HELP**
>
> **HOMEWORK HELP**
> **Example 1:** Exs. 17–25
> **Example 2:** Exs. 26–33
> **Example 3:** Exs. 34–42
> **Example 4:** Exs. 43–50
> **Examples 5, 6:** Exs. 51–65
> **Examples 7, 8:** Exs. 69–73

DOUBLE-ANGLE FORMULAS **Find the exact values of sin 2*x*, cos 2*x*, and tan 2*x*.**

30. $\tan x = 2,\ 0 < x < \frac{\pi}{2}$

31. $\tan x = -\frac{1}{2},\ -\frac{\pi}{2} < x < 0$

32. $\cos x = -\frac{1}{3},\ \pi < x < \frac{3\pi}{2}$

33. $\sin x = -\frac{3}{5},\ \frac{3\pi}{2} < x < 2\pi$

SIMPLIFYING TRIGONOMETRIC EXPRESSIONS **Rewrite the expression without double angles or half angles, given that $0 < x < \frac{\pi}{2}$. Then simplify the expression.**

34. $\sqrt{2 + 2\cos x}\left(\cos \frac{x}{2}\right)$

35. $\frac{\sin 2x}{\sin x}$

36. $\tan 2x\,(1 + \tan x)$

37. $\cos 2x - 3\sin^2 x$

38. $\frac{\cos 2x}{\cos^2 x}$

39. $\left(\frac{\sin x}{1 - \cos^2 x}\right)\tan \frac{x}{2}$

40. $(1 + \cos x)^2 \tan \frac{x}{2}$

41. $\frac{1 + \cos 2x}{\cot x}$

42. $\frac{\sin \frac{x}{2} \tan \frac{x}{2}}{1 - \cos x}$

VERIFYING IDENTITIES **Verify the identity.**

43. $(\sin x + \cos x)^2 = 1 + \sin 2x$

44. $1 + \cos 10x = 2\cos^2 5x$

45. $\cos\theta + 2\sin^2 \frac{\theta}{2} = 1$

46. $\sin \frac{\theta}{3} \cos \frac{\theta}{3} = \frac{1}{2}\sin \frac{2\theta}{3}$

47. $\cos 3x = \cos^3 x - 3\sin^2 x \cos x$

48. $\sin 4\theta = 4\sin\theta\cos\theta\,(1 - 2\sin^2\theta)$

49. $\cos^2 2x - \sin^2 2x = \cos 4x$

50. $\cot\theta + \tan\theta = 2\csc 2\theta$

SOLVING TRIGONOMETRIC EQUATIONS **Solve the equation for $0 \le x < 2\pi$.**

51. $\sin \frac{1}{2}x = -1$

52. $\cos x - \cos \frac{1}{2}x = 0$

53. $\sin 2x \cos x = \sin x$

54. $\cos 2x = -2\cos^2 x$

55. $\tan 2x - \tan x = 0$

56. $\sin \frac{x}{2} + \cos x = 1$

57. $\tan 2x = \frac{\cos 2x}{2}$

58. $\tan \frac{x}{2} = \sin x$

59. $\frac{\cos 2x}{\cos^2 x} = 1$

FINDING GENERAL SOLUTIONS **Find the general solution of the equation.**

60. $\cos 2x = -1$

61. $\sin 2x + \sin x = 0$

62. $\cos 2x - \cos x = 0$

63. $\cos \frac{x}{2} - \sin x = 0$

64. $\sin \frac{x}{2} + \cos x = 0$

65. $\cos 2x = 3\sin x + 2$

66. LOGICAL REASONING Show that the three double-angle formulas for cosine are equivalent.

67. LOGICAL REASONING Show that the two half-angle formulas for tangent are equivalent.

68. *Writing* Use the formula at the top of page 878 to explain why the projection angle that maximizes the distance a projectile travels is $\theta = 45°$ when $h_0 = 0$.

69. **PROJECTILE HEIGHT** Find a formula for the maximum height of an object projected from ground level at speed v and angle θ. To do this, find half of the horizontal distance $\frac{1}{32}v^2 \sin 2\theta$ and then substitute it for x in the general model for the path of a projectile (where $h_0 = 0$) at the top of page 878.

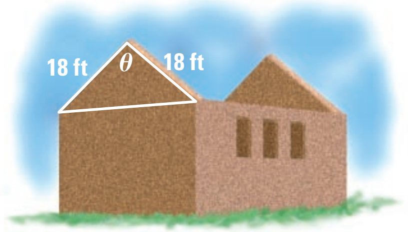
FOCUS ON APPLICATIONS

REAL LIFE **MACHU PICCHU** is an ancient Incan city discovered by Hiram Bingham in 1911 and located in the Andes Mountains near Cuzco, Peru. The site consists of 5 square miles of terraced gardens linked by 3000 steps.

70. **AERONAUTICS** An airplane's mach number M is the ratio of its speed to the speed of sound. When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane (see Lesson 10.5, page 620).

The mach number is related to the apex angle θ of the cone by $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the angle θ that corresponds to a mach number of 4.5.

INCA DWELLING **In Exercises 71 and 72, use the following information.** Shown below is a drawing of an Inca dwelling found in Machu Picchu, about 50 miles northwest of Cuzco, Peru. All that remains of the ancient city today are stone ruins.

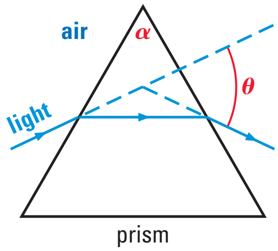


71. Express the area of the triangular portion of the side of the dwelling as a function of $\sin \frac{\theta}{2}$ and $\cos \frac{\theta}{2}$.

72. Express the area found in Exercise 71 as a function of $\sin \theta$. Then solve for θ assuming that the area is 132 square feet.

73. **OPTICS** The index of refraction n of a transparent material is the ratio of the speed of light in a vacuum to the speed of light in the material. Some common materials and their indices are air (1.00), water (1.33), and glass (1.5). Triangular prisms are often used to measure the index of refraction based on this formula:

$$n = \frac{\sin\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)}{\sin \frac{\theta}{2}}$$



For the prism shown, $\alpha = 60°$. Write the index of refraction as a function of $\cot \frac{\theta}{2}$. Then find θ if the prism is made of glass.

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 74–76, choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

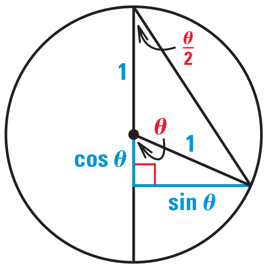
Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
74.	$\sin x$, with $45° < x < 90°$	$\sin 2x$
75.	$\cos x$, with $90° < x < 135°$	$\cos 2x$
76.	$\tan x$, with $45° < x < 90°$	$\tan 2x$

★ Challenge

77. **DERIVING FORMULAS** Use the diagram shown at the right to derive the formulas for $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ when θ is an acute angle.



EXTRA CHALLENGE
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MIXED REVIEW

FUNCTION OPERATIONS **Let $f(x) = 4x + 1$ and $g(x) = 6x$. Perform the indicated operation and state the domain.** (Review 7.3)

78. $f(x) + g(x)$ **79.** $f(x) - g(x)$ **80.** $f(x) \cdot g(x)$

81. $f(x) \div g(x)$ **82.** $f(g(x))$ **83.** $g(f(x))$

CALCULATING PROBABILITIES **Calculate the probability of rolling a die 30 times and getting the given number of 4's.** (Review 12.6)

84. 1 **85.** 3 **86.** 5

87. 6 **88.** 8 **89.** 10

SIMPLIFYING EXPRESSIONS **Simplify the expression.** (Review 14.6)

90. $\cos\left(x - \frac{3\pi}{2}\right)$ **91.** $\sin(x - \pi)$ **92.** $\tan\left(x - \frac{\pi}{3}\right)$

93. $\cos(x - \pi)$ **94.** $\sin\left(x + \frac{\pi}{2}\right)$ **95.** $\tan\left(x + \frac{\pi}{4}\right)$

96. **MOVING** The truck you have rented for moving your furniture has a ramp. If the ramp is 20 feet long and the back of the truck is 3 feet above the ground, at what angle does the ramp meet the ground? (Review 13.4)

QUIZ 3

Self-Test for Lessons 14.6 and 14.7

Find the exact value of the expression. (Lesson 14.6)

1. $\sin 105°$ **2.** $\cos 285°$ **3.** $\tan 165°$

4. $\sin \frac{17\pi}{12}$ **5.** $\cos \frac{13\pi}{12}$ **6.** $\tan\left(-\frac{\pi}{12}\right)$

Given $\sin u = \frac{1}{3}$ with $\frac{\pi}{2} < u < \pi$, find the exact value of the expression. (Lesson 14.7)

7. $\sin \frac{u}{2}$ **8.** $\cos \frac{u}{2}$ **9.** $\tan \frac{u}{2}$ $\frac{\sqrt{3 + 2\sqrt{2}}}{\sqrt{3 - 2\sqrt{2}}}$

10. $\sin 2u$ **11.** $\cos 2u$ **12.** $\tan 2u$

Simplify the expression. (Lessons 14.6, 14.7)

13. $\sin(x + 3\pi)$ **14.** $\cos(\pi - x)$ **15.** $\tan\left(x + \frac{\pi}{4}\right)$

16. $\frac{\sin 2x}{2 \cos x}$ **17.** $2\cos^2 \frac{x}{2} - \cos x$ **18.** $\left(\frac{1 - \tan x}{2}\right) \tan 2x$

Solve the equation. (Lessons 14.6, 14.7)

19. $\sin 3x = 0.5$ **20.** $\cos 2x - \cos^2 x = 0$

21. $\sin(2x - \pi) = \sin x$ **22.** $\tan(-2x) = 1$

23. **GOLF** Use the formula $x = \frac{1}{32}v^2 \sin 2\theta$ to find the horizontal distance x (in feet) that a golf ball will travel when it is hit at an initial speed of 50 feet per second and at an angle of 40°. (Lesson 14.7)