# Testing Learning Task

## Math Goals

- Calculate binomial probabilities and look at binomial distributions
- Graph probability distributions
- Calculate the mean of probability distributions, expected values
- Calculate theoretical and empirical probabilities of probability distributions

**MGSE9-12.S.CP.9** Use permutations and combinations to compute probabilities of compound events and solve problems.

**MGSE9-12.S.MD.1** Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

**MGSE9-12.S.MD.2** Calculate the expected value of a random variable; interpret it as the mean of a probability distribution.

## Standards for Mathematical Practice

**1. Make sense of problems and persevere in solving them.**
**2. Reason abstractly and quantitatively.**
**3. Model with mathematics**
**4. Attend to precision**

## Introduction

In this task students develop an understanding of binomial probability distributions. Students will also compute conditional probabilities where they will distinguish between dependent and independent events. The will graph probability distributions and calculate expected values.

## Materials

- Graphing calculators
- Graph paper

Today you are going to determine how well you would do on a true/false test if you guessed at every answer. Take out a sheet of paper. Type randint(1,2) on your calculator. If you get a 1, write “true.” If you get a 2, write “false.”

Do this 20 times.

Before the teacher calls out the answers, how many do you expect to get correct? Why?

***If you have a small class, have students do this activity twice so you will have a decent sample size.***

***Some should say 10. Ask them why? Since the probability of "true" is ½, they should say that they should get ½ of 20 correct.***

***Teacher should randomly generate answers for the true-false test and call them out.***

Grade your test. How many did you actually get correct? Did you do better or worse than you expected?

1) Make a dot plot of the **class distribution** of the total number correct on your graph paper.

Calculate the mean and median of your distribution. Which measure of center should be used based on the shape of your dot plot?

***Discuss with the students which measure of center should be used based on the shape of the data. The mean should be a good measure because the dot plot is expected to be symmetrical (bell shaped) about 10.***

3) Based on the class distribution, what percentage of students passed?

***Solutions:***
***Answers will depend upon class data. Students should use the dot plot to answer the following questions.***

4) Calculate the probabilities based on the dot plot:

a) What is the probability that a student got less than 5 correct?

b) What is the probability that a student got exactly 10 correct?

c) What is the probability that a student got between 9 and 11 correct (inclusive)?

d) What is the probability that a student got 10 or more correct?

e) What is the probability that a student got 15 or more correct?

f) What is the probability that a student passed the test?

g) Is it more likely to pass or fail a true/false test if you are randomly guessing?

h) Is it unusual to pass a test if you are randomly guessing?

***It's important to note the assumptions for a binomial distribution. They are as follows:***
****Each trial has two outcomes….success "p" or failure "1-p."***
****There is a fixed number of trials "n."***
****The probability of success does not change from trial to trial.***
****The trials are independent. The results for one trial do not depend on the results from another trial.***

For a situation to be considered as having a ***binomial distribution***, the following conditions must be satisfied:

- Each observation/trial has one of **two outcomes**. These two outcomes are referred to as "success" or "failure".
- There are a **fixed number of observations/trials**. The number of observations/trials is referred to as *n*.
- The observations/trials must be **independent.**
- The **probability of success**, referred to as *p*, **is the same** for each observation/trial.

5) Can this true-false test be considered a binomial setting? Why or why not?

***Yes, it meets all of the conditions above***

Binomial Probability

When $X$ has the binomial distribution with $n$ observations and probability $p$ of success on each observation, the possible values of $X$ are 0, 1, 2, . . ., $n$. The probability of $X$ successes in this setting is computed with the formula:

$$P(X)=\binom{n}{r}(p)^X(1-p)^{n-X} \quad \text{or} \quad P(X)=\left(\frac{n!}{(n-X)!X!}\right)(p)^X(1-p)^{n-X}$$

***Students will need you to model so calculate the probability that the student got exactly 5 correct on the test.***

***Explanation: If the student got 5 correct, then 15 were incorrect. P(correct) = ½ and the P(incorrect) = ½. So, if the first 5 were correct, and the last 15 were incorrect, then the student would have CCCCCIIIIIIIIIIIIIII graded on the quiz. The probability of getting that in that order is*** $\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{15}$***. But, the problem did not specify that the first 5 were correct and the last 15 were incorrect. So that leads us to the question, "How many ways can we rearrange the 5 correct and 15 incorrect problems?" I usually remind students of how we rearranged the letters to the word "Mississippi." Then, they usually figure out that the answer is*** $\frac{20!}{5!15!}$ ***since there are five C's that repeat and fifteen I's that repeat. So our final answer to the question, "What is the probability that the student got exactly 5 correct is*** $\frac{20!}{5!15!}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{15}$***. At this point, you can relate*** $\frac{20!}{5!15!}$ ***to*** ${}_{20}C_5$ ***on the calculator.***

****Additional Problem: Calculate the probability that the student got fewer than 3 correct on the test.***

***Explanation: First you need to make sure that they understand that fewer than 3 means 0, 1, or 2 correct. Since these outcomes are mutually exclusive (if you get 0 correct, then you will not get 1 correct, etc.), then you can add the probabilities. So, you would calculate each of the following probabilities:***

***P(0 correct) =*** ${}_{20}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{20}$ ***or*** $\left(\frac{1}{2}\right)^{20}$

***P(1 correct) =*** ${}_{20}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{19}$

***P(2 correct) =*** $_{20}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{18}$

***Once your students have <u>mastered</u> this reasoning and formula, then you may want to show them the calculator buttons "binompdf(n, p, x) and binomcdf(n, p, x)." On the TI-84, you can find "binompdf" under "2nd", "VARS", and then scroll down. If you want to calculate the probability of getting exactly 5 correct on the test, then you would type binompdf(20,1/2,5). If you want to calculate the probability of fewer than 3 are correct, then you would type binomcdf(20, ½, 2). Unlike the binompdf key, the binomcdf key adds up the probabilities starting at 0 and ending at x. The "c" stands for cumulative in binomcdf.***

***Now the students should have enough information to calculate the theoretical probabilities of the following:***

6) Still considering that T-F test, calculate the following probabilities using the Binomial Distribution:

a) What is the probability that a student got less than 5 correct?

***(P(0) + P(1) + P(2) + P(3) +P(4))**** $.5^{20}$ ***or binomcdf(20,.5,4)=.0059***

b) What is the probability that a student got exactly 10 correct?

***P(10)/*** $(2^{20})$ ***or binompdf(20, .5, 10)= .1762***

c) What is the probability that a student got between 9 and 11 correct (inclusive)?

***(P(9) + P(10) + P(11))**** $.5^{20}$ ***or binomcdf(20, .5, 11)- binomcdf(20, .5, 8) =.4966***

d) What is the probability that a student got 10 or more correct?

***(P(10) + P(11) + P(12)+ P(13) + P(15) + P(16) + P(17) + P(18) +P(19) + P(20))/*** $(2^{10})$ ***or***

***1-binomcdf(20, .5, 9) =.5881***

e) What is the probability that a student got 15 or more correct?

***(P(15) + P(16) + P(17) + P(18) +P(19) + P(20))**** $.5^{20}$ ***or***
***1-binomcdf(20, .5, 14)=.0207***

f) What is the probability that a student passed the test?

***(P(14) + P(15) + P(16) + P(17) + P(18) +P(19) + P(20))/*** $2^{10}$ ***or 1-binomcdf(20, .5, 13)=.0577***

***Discuss: How do the theoretical probabilities compare to the experimental probabilities?***

g) Is it more likely to pass or fail a true/false test if you are randomly guessing?

h) Is it unusual to pass a test if you are randomly guessing?

**Testing Learning Task (Part 2)**

Suppose there is a 5 question multiple choice test. Each question has 4 answers (A, B, C, or D).

1) Can this multiple choice test be considered a binomial setting? Why or why not?

***Yes, it meets all of the conditions***

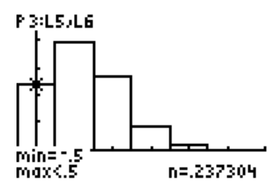
2) If you are strictly guessing, calculate the following probabilities:

a) P(0 correct) = ${}_5C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5$ ***= binompdf(5, .25,0)=.2373***

b) P(1 correct) = ***.3955***
c) P(2 correct) = ***.2637***
d) P(3 correct) = ***.0879***
e) P(4 correct) = ***.0146***
f) P(5 correct) = ***.0010***

3) Draw a histogram of the probability distribution for the number of correct answers on graph paper. Label the *x-axis* as the **number of correct answers**. The *y-axis* should be the **probability of x**.

***The histogram should be skewed right as shown in this example.***




4) Based on the distribution, how many problems do you expect to get correct?

***Comment: the mean of a binomial distribution is np or 5(.25) = 1.25. From the graph, students should say that they expect one correct answer if guessing.***

5) Based on the distribution, how likely is it that you would pass if you were strictly guessing? (*Calculate the probability of getting 4 or 5 correct.)*

***P(4 or 5 correct) = .0146+.0010= .0156 So, it is not likely that you would pass by guessing alone.***

6) What is the probability that you will get less than 3 correct? ***.8965***

7) What is the probability that you will get at least 3 correct? ***0.1035***

Now let's look at tests, such as the SAT, when you are penalized for guessing incorrectly. Suppose you have a multiple choice test with five answers (A, B, C, D, or E) per problem. The probability your guess is **correct = 1/5** and the probability that your guess is **incorrect = 4/5**.

Suppose the test that you are taking will award you one point for each question correct, but penalize you by ¼ of a point for each question you answer incorrectly. Test scores will be rounded to the nearest 10 percent.

8) If you strictly guess and get exactly 4 correct and 6 incorrect, what would be your score?

***4 – (.25)6 = 2.5, so your rounded score would be 3/10 or 30%***

9) If you take a 10 question test and know that 8 questions are correct, should you guess the answers for the other two questions?

***8 – 0 .25(2) = 7.5 which rounds to 8. Yes, you have nothing to lose.***

10) If you take a 10 question test and know that 6 questions are correct, should you guess the answers for the other 4 questions?

***6 – 0.25(4) = 5. You could make a 50%; however, if you guessed one of the 4 correctly (which is expected), then you would still make a 60%.***

11) Given that you answered all 10 questions and you knew that 6 were correct, answer the following questions:

a) If you can eliminate one of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

***The probability of guessing correctly for the 4 unknown problems is 1/4 since one answer is eliminated from 5 possible answers. Therefore, from the 4 questions which you did not know the answer, you expect 4(1/4) = 1 to be correct and 4(3/4)=3 to be incorrect. So, your expected score should be 6 + 1 – (.25)(3) = 6.25 which rounds to 6 (or 60%).***

b) If you can eliminate two of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

***6+ 4(1/3) – 4(2/3)(.25) = 6.666 which rounds to 7 (or 70 %)***

c) If you can eliminate three of the answer choices for each of the 4 questions for which you are guessing, what would your expected score be?

***6 + 4(1/2)-4(1/2)(.25)=7.5 which rounds to 8 (or 80%)***

## Testing Learning Task (Part 3)

Earlier, we found the probability that the student passed a multiple choice test just by random guessing. However, we know that students usually have a little more knowledge than that, even when they do not study, and consequently do not guess for all problems.

Suppose that a student can retain about 30% of the information from class without doing any type of homework or studying. If the student is given a 15 question multiple choice test where each question has 4 answer choices (A, B, C, or D), then answer the following questions:

1. What is the probability that the student gives the correct answer on the test? What would be her percentage score on a 15 question test?

2. Given she provides the correct answer on the test, what is the probability that she strictly guessed?

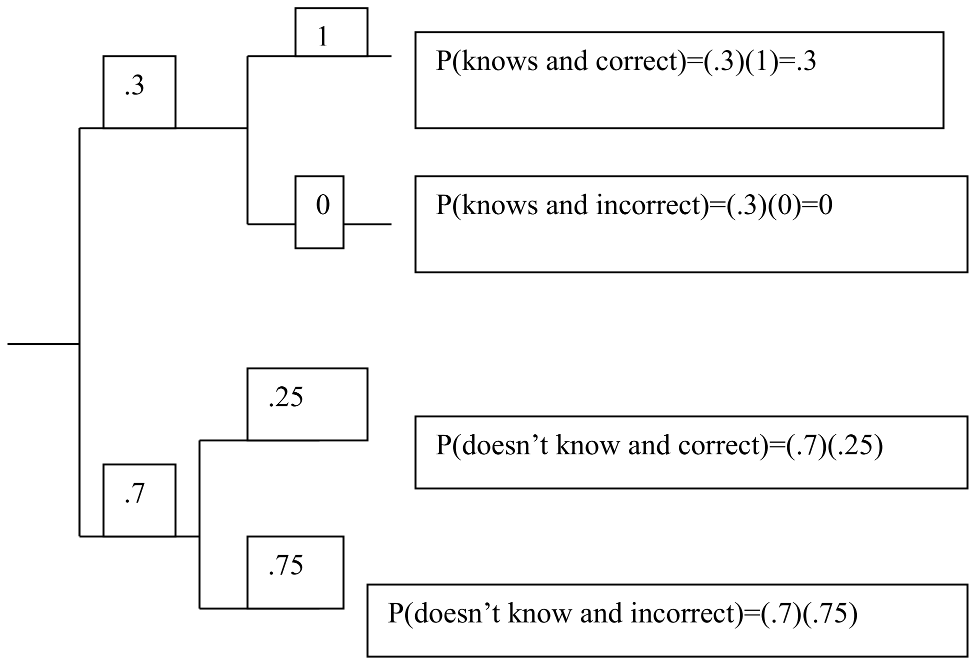
***You may need to use the formula for conditional probability to do this problem. The formula is as follows:***

***P(A given B) = P(A and B)/P(B). It's symbolically written as such*** $P(A|B) = \frac{P(A \cap B)}{P(B)}$

***where*** $\cap$ ***stands for the intersection of sets A and B.***

***Note: By solving the above formula,*** $P(A \cap B) = P(A|B) \cdot P(B)$.

***With conditional probability problems, I have found tree diagrams very helpful. To solve the problem, I would make a tree diagram. The first branches would be P(knows answer) and P(does not know answer). The next branches would be P(correct given she knows answer), P(incorrect given she knows the answer), P(correct given she does not know the answer), and P(incorrect given she does not know the answer). The product of the two branches would be P(A and B)…P(knows answer and correct), P(knows answer and incorrect), P(doesn't know answer and correct), P(doesn't know answer and incorrect). The tree diagram should look like the one below.***




***Using the tree diagram, and conditional probability formula, we can answer the following questions:***

***What is the probability that the student gives the correct answer on the test? The student can give the correct answer when she knows it or when she doesn't and guesses correctly. So the answer is (.3)(1) + (.7)(.25) = .475***

***What would be her percentage score on a 15 question test? 47.5%***

***Given she provides the correct answer on the test, what is the probability that she strictly guessed?***
***This is a conditional probability problem. You would use the following formula to solve the problem:*** $P(guess \mid correct) = \frac{P(guess\ and\ correct)}{P(correct)}$

***From the tree diagram*** $P(guess\ and\ correct)$ ***= (.7)(.25)= .175. Note: she would not guess if she knew the answer. We already calculated*** $P(correct)$ ***=.475.***

***Therefore,*** $P(guess \mid correct) = \frac{.175}{.475} \approx .368.$

## Testing Learning Task

Name ______________________________________________ Date______________

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**MGSE9-12.S.MD.1** Define a random variable for a quantity of interest by assigning a numerical value t each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

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